PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 3, November 1985

CONTINUOUS FUNCTIONS ON THE SPACE OF PROBABILITIES

S. C. BAGCHI AND B. V. RAO

ABSTRACT. Weiss and Dubins discovered that any continuous function $g(P)$ on the space of probabilities $\mathscr{P}$ of a compact Hausdorff space K is of the form $\int f\, dP^{\infty}$ for some continuous function f on K^{∞}. A short proof is given here.

Let $\mathscr{P}$ be the space of probabilities on the Borel σ field of the unit interval $K = [0,1]$ equipped with the weak* topology so that $\mathscr{P}$ is again compact. It is easy to see that if f is a real continuous function on K^{∞} then $g(P) = \int f\, dP^{\infty}$ is a continuous function on $\mathscr{P}$. Recently B. Weiss and L. E. Dubins [**1**] proved the converse. Namely,

THEOREM. *Given a real continuous function g on $\mathscr{P}$ there is a real continuous function f on K^{∞} such that $g(P) = \int f\, dP^{\infty}$ for all P in $\mathscr{P}$.*

For $f \in C(K^{\infty})$, let the function g defined above be denoted by Tf. T is obviously a continuous linear operator on $C(K^{\infty})$ into $C(\mathscr{P})$. Observe that if f_1, f_2 are in $C(K^{\infty})$ then f defined on K^{∞} by

$$f(x_1, x_2, \ldots) = f_1(x_1, x_3, \ldots) f_2(x_2, x_4, \ldots)$$

is continuous and $Tf = Tf_1 \cdot Tf_2$. As a consequence range of T is a subalgebra of $C(\mathscr{P})$ containing constants and it clearly separates points. We shall now show that T^* has closed range. This, by Banach's closed range theorem [**2**] implies that range of T is closed and, hence, by the Stone-Weierstrass theorem, must be all of $C(\mathscr{P})$ as claimed.

Observe that $T^*\colon \mathfrak{M}(\mathscr{P}) \to \mathfrak{M}(K^{\infty})$ is given by

$$T^*\mu(A) = \int P^{\infty}(A)\, d\mu(P) \quad \text{for } \mu \in \mathfrak{M}(\mathscr{P}).$$

$\mathfrak{M}(X)$ is the space of finite signed measures on X with total variation norm. As T has dense range, T^* is one-to-one. If μ is positive then so is $T^*\mu$. If μ_1, μ_2 are positive and orthogonal then so are $T^*\mu_1$ and $T^*\mu_2$. Indeed, if $L \subset \mathscr{P}$ is a Borel set such that μ_1 sits on L and μ_2 sits on L^c, then a simple application of the Glivenko-Cantelli Lemma implies that $T^*\mu_1$ sits on A and $T^*\mu_2$ sits on A^c where

$$A = \left\{ (x_1, x_2, \ldots) \in K^{\infty} \colon \frac{1}{n} \sum_1^n \delta_{x_i} \text{ has a limit in } \mathscr{P} \text{ and that limit} \in L \right\}.$$

Received by the editors May 31, 1984.
1980 *Mathematics Subject Classification*. Primary 60B99.
Key words and phrases. Weak* convergence of probabilities, closed range theorem.

As usual δ_x is the point mass at x. This implies that if $\mu = \mu^+ - \mu^-$ is the canonical Jordan Hahn decomposition of $\mu \in \mathfrak{M}(\mathscr{P})$ then $T^*\mu^+ - T^*\mu^-$ is the canonical decomposition of $T^*\mu$. This in turn implies that the total variations of μ and $T^*\mu$ are equal. In other words, T^* is a norm preserving map and hence Range T^* is closed. This completes the proof.

REMARK 1. The same proof applies if K is any compact Hausdorff space and $\mathscr{P}$ the probabilities on the Baire σ field. This is the form proved in [**1**].

REMARK 2. Dr. G. Jogesh Babu has yet another proof of the Theorem patterned after the original proof in [**1**] but more probabilistic in nature.

REMARK 3. The representation theorem of Hewitt-Savage for symmetric probabilities identifies range of T^* as precisely the linear span of symmetric probabilities.

REFERENCES

1. L. E. Dubins, *Bernstein like polynomial approximation in higher dimensions*, Pacific J. Math **109** (1983), 305–311.

2. K. Yosida, *Functional analysis*, Springer-Verlag, Berlin and New York, 1974.

STATISTICS AND MATHEMATICS DIVISION, INDIAN STATISTICAL INSTITUTE, 203 BARRACKPORE TRUNK ROAD, CALCUTTA 700 035, INDIA