LIMIT THEOREMS FOR DIVISOR DISTRIBUTIONS

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ABSTRACT. For a positive integer $N$, let $X_N$ be a random variable uniformly distributed over the set $\{\log d \mid d \mid N\}$. Let $F_N$ be the normalized (to have expectation zero and variance one) distribution function for $X_N$. Necessary and sufficient conditions for the convergence of a sequence $F_N$ of distributions are given. The possible limit distributions are investigated, and the case where the limit distribution is normal is considered in detail.

1. Introduction. Let the positive integer $N$ have prime factorization $N = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Define $\mu_n(N)$, for positive integer $n$, by

$$\mu_n = \left(12^{-1} \sum_{1 \leq j \leq k} ((\alpha_j + 1)^n - 1)(\log p_j)^n\right)^{1/n}.$$

The divisor distribution of $N$ refers to the function

$$F_N(x) = \tau^{-1} \sum_{d \mid N} \frac{1}{d}$$

where $\tau$ is the number of divisors of $N$, and the sum is restricted to those divisors satisfying $\log(d/\sqrt{N}) \leq x\mu_2$.

In this paper, we determine when a sequence of divisor distributions tends to a limit, and investigate the limit distributions that arise. Erdős and Nicolas [2] had previously shown the divisor distribution of $N_j = \prod_{p < j} p$ (we reserve the letters $p$ and $q$ for primes) to be asymptotically normal as $j \to \infty$. With regard to the normal distribution we prove

THEOREM 2. The normal distribution

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp\left(-\frac{1}{2} t^2\right) dt$$

is the only infinitely divisible distribution that can arise as the limit of a sequence $F_{N_j}$ of divisor distributions. A necessary and sufficient condition for convergence is that

$$\lim_{j \to \infty} \left(\mu_2(N_j)\right)^{-1} \mu_\infty(N_j) = 0.$$

Moreover,

$$\sup_w |F_{N_j}(w) - \Phi(w)| \ll \frac{\mu_{\infty}}{\mu_2}.$$

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and

\[ \left| \frac{1 - F_N(x)}{1 - \Phi(x)} - 1 \right| \ll \left( x + \frac{1}{x} \right) \frac{\mu_4}{\mu_2} \]

for \( x \leq \mu^{-1} \).

Here \( \mu_\infty(N) \) means \( \lim_{n \to \infty} \mu_n(N) \).

We define the (Fourier) transform of a distribution \( F \) as

\[ \hat{F}(t) = \int_R e^{2\pi i tx} dF(x). \]

If \( \hat{F}(t) \) is the restriction to \( R \) of an entire function, we say that \( \hat{F} \) is entire. In the general case we have

**Theorem 1.** A necessary and sufficient condition for a sequence \( F_N \) of divisor distributions to converge to a distribution \( F \) is that for each \( n \) the limits \( a_n = \lim_{j \to \infty} \mu_{2n}(N_j)(\mu_2(N_j))^{-1} \) exist. In this case \( \hat{F} \) is entire and is represented in the disk \( |z| < 1/4 \) by

\[ \hat{F}(z) = \exp \left( -\sum_{n=1}^{\infty} (n(2n)!)^{-1} 6B_n(2\pi a_n z)^{2n} \right), \]

where

\[ B_n = 4n \int_0^{\infty} \left( e^{2\pi t} - 1 \right)^{-1} t^{2n-1} dt \]

are the Bernoulli numbers.

We say that a sequence \( F_j \) of distributions converges to \( F \), if \( F_j(\pm \infty) \to F(\pm \infty) \), and \( F_j(x) \to F(x) \) at all continuity points \( x \) of \( F \).

A reasonable characterization of the possible limit distributions seems difficult. We do however have the following “factoring theorem”:

**Theorem 3.** Suppose the sequence \( F_N \) of divisor distributions converges to \( F \). If \( F \) is not a finite point mass distribution then, for some \( \phi \in [0, \pi/2) \),

\[ F(x) = G(x \sec \phi) * H(x \csc \phi). \]

The convolution factor \( G \) is a normal, uniform, or singular distribution. \( H \) is the limit of a sequence of divisor distributions when \( \phi > 0 \), and otherwise is to be interpreted as point mass at 0. Moreover, if \( \liminf_{j \to \infty} \omega(N_j) = K < \infty \), then \( F \) may be written as a convolution product involving no more than \( K \) uniform or arithmetic distributions.

Here \( \omega(N) \) is the number of distinct prime divisors of \( N \). A finite point mass distribution is a finite convolution of arithmetic distributions. An arithmetic distribution is a probability distribution with zero expectation, and, a step function, whose finitely many jump discontinuities are of equal height and occur along an arithmetic progression. A uniform distribution has density \((12)^{-1/2} \chi_{[-\sqrt{3}, \sqrt{3}]} \) (where \( \chi_I \) is the indicator of the interval \( I \)), and a singular distribution is continuous with zero derivative almost everywhere.
2. Necessary and sufficient conditions.

**Lemma 1.** The transform $\hat{F}_N$ of the divisor distribution of $N$ is an entire function which is represented in the disk $|z| < \mu_2 \mu_{\infty}^{-1}$ by

$$\hat{F}_N(z) = \exp\left(-\sum_{n=1}^{\infty} (n(2n)!)^{-1}B_n\left(\mu_2 \mu_{\infty}^{-1} 2\pi z\right)^2\right).$$

**Proof.** $\hat{F}_N$ is entire since $dF_N$ is compactly supported. For notational convenience, let $u = \pi \mu_{\infty}^{-1} t$. The transform of $F_n$ is

$$F(t) = \prod_{j=1}^{k} \left((\alpha_j + 1) \sin(u \log p_j)\right)^{-1} \sin(u (\alpha_j + 1) \log p_j).$$

Taking the logarithm of (2.1) yields

$$\hat{F}(t) = \exp\left(\sum_{j=1}^{k} \int_{0}^{\infty} u(u_j + 1) \log p_j \right) \cot x - x^{-1} dx \right).$$

Substituting the power series for $\cot x - x^{-1}$ in (2.2), integrating term by term, and interchanging the order of summation completes the proof. \(\square\)

**Corollary 1.** If $|y| < \lambda < 1/4$, then $F_N(x + iy) \ll 1$. Also, $\hat{F}$ has a zero at $\mu_2 \mu_{\infty}^{-1}$.

**Proof.** (2.1) shows that $\hat{F}_N(\mu_2 \mu_{\infty}^{-1}) = 0$. If $m$ is a positive integer, $m < 2n$, then

$$\mu_2 \mu_{\infty}^{-1} \leq (1 - 2^{-m})^{-1/m} 12^{1/m - 1/2}.$$ 

Since $\hat{F}_N(x + iy) \ll \hat{F}_N(0)$, using inequality (2.3) in Lemma 1 completes the proof.

**Proof of Theorem 1.** By Corollary 1, the collection $\mathcal{F} = \{\hat{F}_{N_j}\}_{j=1}^{\infty}$ of analytic functions is uniformly bounded on compact subsets of $G_\lambda = \{x + iy \in \mathbb{C}: |y| < \lambda < 1/4\}$. Therefore, Montel's theorem \([1]\) implies that any sequence of functions from $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets of $G_\lambda$. This, together with the representation of $\hat{F}_N$ provided by Lemma 1, implies that the sequence $\hat{F}_{N_j}$ converges to a function $H$ if and only if the limits $a_n$ exist, and in such case,

$$H(z) = \exp\left(-\sum_{n=1}^{\infty} (n(2n)!)^{-1}B_n\left(a_n 2\pi z\right)^2\right).$$

in the disk $|z| < 1/4$. It follows from the continuity theorem for Fourier-Stieltjes transforms that the convergence of $\hat{F}_{N_j}$ to such a function $H$ is equivalent to the convergence of $F_{N_j}$ to some distribution $F$, and in such case, $\hat{F} = H$. It remains to show that $\hat{F}$ is entire.

The inequality (for $x \geq 0$)

$$1 - F(x) \leq F(-x) \leq \exp\{-x^2/6\}$$

implies that the sequence of entire functions

$$H_L(z) = \int_{-L}^{L} e^{2\pi izx} dF(x)$$


is uniformly Cauchy on compact subsets of \( \mathbb{C} \). It therefore suffices to prove (2.4). Note that for positive \( \lambda \) and positive \( x \),

\[
1 - F_N(x) \leq F_N(-x) \leq \tau(N)^{-1} \sum_{d|N} \exp \left( -\lambda x + \lambda \mu_2^{-1} \log \frac{\sqrt{N}}{d} \right).
\]

With \( \lambda_j = \lambda \mu_2^{-1} \log p_j \), the right-hand side of (2.5) is equal to

\[
e^{-\lambda \chi} \prod_{j=1}^k \left( \exp \left( \frac{1}{2} a_j \chi \right) \left( \alpha_j + 1 \right)^{-1} \sum_{n=0}^{\alpha_j} \exp \left( -n \lambda \chi \right) \right).
\]

Using the convexity of \( e^x \) and the inequality \( \cosh(x) \leq \exp(x^2/2) \), we see that (2.6) is not greater than \( \exp(-\lambda x + \lambda^2/2) \). Choosing \( \lambda = 3^{-1/4} \) finishes the proof.

Note that our method of proving Theorem 1 (via Montel’s Theorem) shows that any sequence of \( F_{N_j} \) (or \( \hat{F}_{N_j} \)) has a subsequence which converges to some \( F \) (respectively \( \hat{F} \)).

**Proof of Theorem 2.** Suppose \( F_{N_j} \) converges to \( \Phi \). By Theorem 1, we have

\[
\Phi(t) = \exp \left\{ - \sum_{n=1}^{\infty} \left( n (2n)! \right)^{-1} 6 B_n \left( a_n 2\pi t \right)^{2n} \right\}.
\]

On the other hand, \( \hat{\Phi}(t) = \exp\{-2\pi^2 t^2\} \). It follows that \( a_j = 0 \) for \( j > 1 \). Conversely, if \( a_j = 0 \) for \( j > 1 \), then the sequence \( F_{N_j} \) converges to some distribution \( F \), where \( \hat{F}(t) = \exp\{-2\pi^2 t^2\} \). Hence \( F = \Phi \). It follows that \( \mu_\infty \mu_2^{-1} \to 0 \) is necessary and sufficient since

\[
\mu_\infty \mu_2^{-1} \ll \mu_2 \mu_2^{-1} \ll \left( \mu_\infty \mu_2^{-1} \right)^{1/4}.
\]

If \( F_{N_j} \) does not converge to \( \Phi \), then there is some compact interval of \( \mathbb{R} \) containing infinitely many of the points \( t_j = \mu_2 \left( N_j, \left( \mu_\infty(N_j)^{-1} \right) \right. \). Let \( t^* \) be a limit point. Each \( t_j \) is a zero of \( \hat{F}_{N_j} \) by Corollary 1, so if \( F_{N_j} \) were to converge to \( F \), then \( F(t^*) = 0 \). This precludes the possibility that \( F \) is infinitely divisible, since such distributions have positive transforms.

The first inequality of Theorem 2 is a straightforward application of the following result, referred to as the Berry–Eseen inequality. Let \( F \) and \( G \) be probability distributions, and suppose \( G \) has density \( g \). Then for all \( T > 0 \)

\[
\sup_x |F(x) - G(x)| \ll T^{-1/2} \|g\|_\infty + \int_{-T}^T |t|^{-1} |\hat{F}(t) - \hat{G}(t)| dt
\]

(Feller [3]).

To prove the second inequality of Theorem 2, define the measures \( dV \) and \( dG \) by

\[
dV(x) = e^{-A + yx} dF_N(x), \quad dG(x) = (2\pi)^{-1/2} e^{-\left( x - y \right)^2/2} dx.
\]

Let \( R(x) = \exp(x^2/2)(1 - \Phi(x)) \). Then

\[
\frac{1 - F_N(y)}{1 - \Phi(y)} = e^{y^2/2} R(y)^{-1} \left( I + e^{A - y^2} R(y) \right),
\]

where

\[
I = e^A \int_0^\infty e^{-yx} d \left( V(x) - G(x) \right).
\]
Now define $A - y^2/2$ to be the function
\[
H(y) = -\sum_{n=2}^{\infty} 6B_n(n(2n)!)^{-1}(\mu_{2n}\mu_{y}^{-1}y)^{2n}.
\]

Assuming that $|V(x) - G(x)| \leq \Delta$ and $y > 0$, (2.7) becomes
\[
(2.8) \quad \frac{1 - F_N(y)}{1 - \Phi(y)} = e^{H(y)} + O\left(\Delta R(y) e^{H(y)}\right).
\]

It is well known that $R(y)^{-1} \leq \sqrt{2\pi} (y + y^{-1})$ (see for example Mitrinovic [4]), so the proof is completed by establishing, for $0 < y \leq \mu_2/\mu_4$, the inequalities
\[
(2.9) \quad |H(y)| \ll y^4(\mu_4/\mu_2)^4,
\]
and
\[
(2.10) \quad \Delta \ll \mu_4/\mu_2.
\]

$H(m)$ is the sum of an alternating decreasing sequence, hence (2.9). The Berry–Esseen inequality, with $F = V$ and $G = G$, yields (2.10). $\square$

3. The factoring theorem.

**Lemma 3.** If $M$ and $N$ are relatively prime positive integers, then
\[
\hat{F}_{MN}(t) = \hat{F}_M(t\cos \phi) \hat{F}_N(t\sin \phi)
\]
where
\[
\cos \phi = \frac{\mu_2(M)}{\sqrt{(\mu_2(M))^2 + (\mu_2(N))^2}}.
\]

**Proof.** The functions $(\mu_n(N))^n$ are additive. Therefore, Lemma 1 implies
\[
\hat{F}_{MN}(t\mu_2(MN)) = \hat{F}_M(t\mu_2(M)) \hat{F}_N(t\mu_2(N)),
\]
and Lemma 3 follows.

**Lemma 4.** Let $p$ and $q$ be primes, and $\alpha$ a positive integer. Then $F_{p^\alpha} = F_{q^\alpha}$ is an arithmetic distribution with discontinuities at the points
\[
\left\{(2\alpha^{-1}k - 1)((\alpha + 2)^{-1}3\alpha)^{1/2}\right\}_{k=0}^\alpha.
\]

As $\alpha \to \infty$, $F_{p^\alpha}$ converges to the uniform distribution $U$ having density $(12)^{-1/2} \chi_{[-\sqrt{3}, \sqrt{3}]}$.

Lemma 4 follows immediately from Lemma 1 and Theorem 1.

**Lemma 5.** Let $F_j$ be a sequence of arithmetic distributions with $d_j > 1$ discontinuities such that $dF_j$ is supported in $[-1, 1]$. Let $s_j$ be the distance between discontinuities of $F_j$, and assume $v_j$ is a sequence of positive numbers such that
\[
(1) \quad \sum_{j > J} v_j < \frac{1}{4}s_j v_j \text{ for } J = 1, 2, \ldots,
\]
\[
(2) \quad (\prod_{j=1}^{J} d_j) \sum_{j > J} v_j \to 0 \text{ as } J \to \infty.
\]

Then the convolution $H_k(x) = F(x/v_1) * \cdots * F_k(x/v_k)$ converges to a singular distribution as $k \to \infty$.

The proof is easy, and will be omitted.

We now prove Theorem 3. Let $N_j$ have prime factorization
\[
N_j = p_j(1)^{a_j(1)} \cdots p_j(k_j)^{a_j(k_j)}.
\]
We abbreviate \( p_j(i)^{\alpha_j(i)} \) as \( (j, i) \), and use \( \langle x \rangle \) to mean \( 2^x \).

First consider the case \( \lim \inf_{j \to \infty} \omega(N_j) = K \). By passing to a subsequence and reindexing, we may assume \( \omega(N_j) = k \), and \( (j, k) > (j, l) \) for \( k < l \). Let \( v_j(k) = \mu_2((j, k))(\mu_2(N_j))^{-1} \), and \( M_j(k) = (j, k)^{-1}N_j \). Assume that \( F \) is not a finite point mass distribution.

Repeated use of Lemma 3 gives

\[
(3.1) \quad F_{N_j}(t) = \prod_{k=1}^{K} \hat{F}_{(j,k)}(v_j(k)t).
\]

Since each \( v_j(k) \in [0,1] \), we may pass to a subsequence and assume \( v_j(k) \to v_k \) as \( j \to \infty \). If any \( v_k = 0 \), then \( \hat{F}_{(j,k)}(v_j(k)t) \to 1 \) for all \( t \). Hence such a factor can be ignored when considering \( \lim_{j \to \infty} \hat{F}_{N_j} \), and so we may assume \( v_k > 0 \). We may also pass to a subsequence and assume each \( F_{(j,k)} \) in (3.1) converges. Therefore, Theorem 1 implies that either \( \alpha_j(k) \to \infty \) or the sequence \( \alpha_j(k) \) becomes constant, say \( \alpha_j(k) = \alpha_k \), for large \( j \). If for all \( k \), \( \alpha_j(k) \to \alpha_k \), then (3.1) and Lemma 4 give

\[
\hat{F}_{N_j}(t) \to \hat{F}_{(\alpha_k)}(v_{\alpha_k}t) \ldots \hat{F}_{(\alpha_k)}(v_{\alpha_k}t),
\]

so that \( F \) would be a finite point mass distribution, contrary to hypothesis. Therefore, let \( k \) be such that \( \alpha_j(k) \to \infty \), and let \( M_j = M_j(k) \).

By Lemma 3 we have

\[
\hat{F}_{N_j}(t) = \hat{F}_{M_j}(t \sin \phi_j) \hat{F}_{(j,k)}(t \cos \phi_j),
\]

where \( \cos \phi_j = v_j(k) \). As \( j \to \infty \), we have \( \cos \phi_j \to \cos \phi = v_k > 0 \), and by Lemma 4, \( \hat{F}_{(j,k)}(t) \to \hat{U}(t) \). Passing to a subsequence, we have also \( \hat{F}_{M_j} \to \hat{H} \) as \( j \to \infty \).

Note that \( \omega(M_j) < k = \omega(N_j) \); so, by redefining \( N_j \) as \( M_j \), the above argument can be repeated at most \( k - 1 \) times.

Now consider the case \( \omega(N_j) \to \infty \). Assume that \( F \) has no uniform or normal convolution factors, and is not a finite point mass distribution. We will show that \( F \) either has a singular convolution factor, or is the limit of a sequence \( F_L \), with \( \omega(L_j) = O(1) \).

Since \( F \) is not normal, Theorem 2 gives the existence of a \( \delta > 0 \) such that, for infinitely many \( j \), \( \mu_\infty(N_j)(\mu_2(N_j))^{-1} > \delta \). Passing to a subsequence we may assume this for all \( j \). Lemma 3 gives

\[
(3.2) \quad \hat{F}_{N_j}(t) = \hat{F}_{M_j(1)}(t \sin \phi_j) \hat{F}_{(j,1)}(t \cos \phi_j),
\]

where \( \cos \phi_j = v_j(1) \). Inequality (2.3) implies that \( v_j(1) > \delta/4 \), so we may pass to a subsequence and assume \( \cos \phi_j \to \cos \phi = v_1 \geq \delta/4 \) as \( j \to \infty \). If \( \sin \phi = 0 \), then \( \hat{F}_{M_j(1)}(t \sin \phi_j) \to 1 \) for all \( t \). Hence, this factor could be ignored when considering \( \lim_{j \to \infty} F_{N_j} \), and \( F \) would be the limit of a sequence \( F_{L_j} \) with \( \omega(L_j) = O(1) \) (take \( L_j = (j, 1) \)). By passing to a subsequence, we may assume that each factor in (3.2) converges. Since \( F \) has no uniform convolution factor, this implies that the sequence \( \alpha_j(1) \) becomes constant, say \( \alpha_j(1) = \alpha_1 \), for large \( j \).

If we assume that \( F \) is not the limit of a sequence \( F_{L_j} \) with \( \omega(L_j) = O(1) \), then by redefining \( N_j \) as \( M_j(1) \), the above argument can be repeated indefinitely. The \( k \)th application of the argument produces a subsequence \( \hat{F}_{N_{k+1}}, \hat{F}_{N_{k+2}}, \ldots \) of the sequence
generated at the $k - 1$st stage along which

$$\hat{F}_{(j,k)}(v_{j}(k)t) \to \hat{F}_{(a_{j})}(v_{k}t).$$

Let $N_{j} = N_{j1}$ be the diagonal sequence. It follows that

$$\hat{F}_{N_{j}}(t) = \prod_{k=1}^{k_{j}} \hat{F}_{(a_{k1})}(v_{j}(k)t),$$

where for any $k$, $v_{j}(k) \to v_{k}$ and $a_{j}(k) \to a_{k}$ as $j \to \infty$.

Fatou’s Lemma gives

$$\sum_{k=1}^{\infty} v_{k}^{2} \leq \liminf_{j \to \infty} \sum_{k=1}^{k_{j}} v_{j}^{2}(k) = 1.$$  

Hence there exists a subset $\{v_{k^{*}}\}_{k^{*}=1}^{\infty}$ of the set $\{v_{k}\}_{k=1}^{\infty}$ satisfying the conditions of Lemma 5 with respect to the distributions $F_{(a_{k^{*}})}(x)$. Let $f: \mathbb{Z}^{+} \to \mathbb{Z}^{+}$ be a nondecreasing function satisfying the following conditions:

(A) $f(j) \leq k_{j}$ and $\lim_{j \to \infty} f(j) = \infty$,

(B) $k < f(j)$ implies $a_{j}(k^{*}) = a_{k}$ and $|v_{j}(k^{*}) - v_{k^{*}}| < 2^{-f(j)}v_{k^{*}}$.

Let $M_{j} = \prod_{k < f(j)}(j, k^{*})$, and $\cos \phi_{j} = (\mu_{2}(N_{j}))^{-1}\mu_{2}(M_{j})$. Applying Lemma 3, we reorganize (3.3) as

$$\hat{F}_{N_{j}}(t) = \hat{F}_{M_{j}}(t \sin \phi_{j}) \prod_{k < f(j)} F_{(a_{k^{*}})}(v_{j}(k^{*})t).$$

By passing to a subsequence, the first factor on the right-hand side of (3.4) converges to $\hat{H}(t \sin \phi)$ for some distribution $H$. The proof is completed by noting that the second factor converges to

$$\prod_{k=1}^{\infty} F_{(a_{k^{*}})}(v_{k^{*}}t),$$

which by Lemma 5 is the transform of a singular distribution.  \(\square\)

**References**


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