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# WEIGHT SPACES OF LIE ALGEBRA MODULES

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ABSTRACT. Let $V$ be a finite-dimensional module for the finite-dimensional Lie algebra $L$ over a field of characteristic zero. If $V^\lambda = \{v \in V |$ all $x \in L,\ [x - \lambda(x)]^i v = 0$ for some $i\}$ is nonzero, then $\lambda \in L^*$ and is a character of $L$. Moreover, the corresponding eigenspace $\{v \in V |$ all $x \in L,\ xv = \lambda(x)v\}$ is nonzero and $V^\lambda$ is an $L$ submodule of $V$.

Let $L$ be a finite-dimensional Lie algebra over the field $k$, $V$ a finite-dimensional $L$ module. If $\lambda$ is a function from $L$ to $k$, define the eigenspace $V_\lambda$ and weight space $V^\lambda$ by

$$V_\lambda = \{v \in V | xv = \lambda(x)v \text{ for all } x \in L\},$$
$$V^\lambda = \{v \in V | \text{ for all } x \in L,\ [x - \lambda(x)]^i v = 0 \text{ for some } i = i(x)\}.$$

It is clear that $V_\lambda$ is a submodule of $V$ and that if $V_\lambda \neq 0$, then $\lambda$ must be linear and in fact a character of $L$ (i.e., a homomorphism from $L$ to $k$; equivalently, $\lambda$ is linear and $\lambda([L, L]) = 0$). In [**5**, Theorem 42] it was asserted that, if $k$ has characteristic zero, then $V^\lambda \neq 0$ implies $\lambda$ is linear. However, the proof tacitly assumed $L$ solvable. In [**7**], the further questions were raised: if $V^\lambda$ is nonzero, must $\lambda$ be a character, and must $V_\lambda$ be nonzero? One may also ask whether $V^\lambda$ must be a submodule. These questions appear to have been answered completely only for $L$ nilpotent [**4**, §II.4]. In this note we answer all three questions affirmatively for arbitrary $L$ when $k$ is of characteristic zero.

THEOREM. *Let $k$ have characteristic zero. Suppose $V^\lambda \neq 0$. Then*
(a) *every semisimple subalgebra of $L$ acts trivially on $V^\lambda$,*
(b) *$\lambda$ is a character,*
(c) *$V^\lambda$ is a submodule of $V$,*
(d) *$V_\lambda \neq 0$.*

PROOF. Note that unless $\lambda$ is known to be linear, there is no natural way to preserve the hypothesis under extension of scalars ($\lambda$ might not extend!).

(a) We will use the fact that a semisimple Lie algebra of endomorphisms on a finite-dimensional vector space is spanned by its semisimple elements. One way to see this is to note first that it is true for the standard two-dimensional representation of the split three-dimensional algebra, since if $e$ and $f$ correspond to $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, then $e + f$, $e + 4f$, and $[e, f]$ act semisimply and span the algebra. Since every irreducible representation of this algebra occurs in the symmetric algebra of this representation, hence has the desired property, the assertion now follows from [**4**, Chapter III, Theorem 17].

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If $x$ acts semisimply, then each $v \in V^\lambda$ is an eigenvector for $x$. Thus if $L$ is semisimple, each $v \in V^\lambda$ is a common eigenvector for $L$, so $\lambda$ must be a character. Since $L = [L, L]$, $\lambda = 0$. Thus $L$ acts trivially on $V^\lambda$.

(b) Suppose first that $L$ is solvable. Let $K$ be the algebraic closure of $k$, $L_K = L \otimes_k K$, $V_K = V \otimes_k K$. Then $L_K$ is solvable and acts in the natural way on $V_K$, so by Lie's Theorem [**5**, Theorem 2.6 or **3**, Corollary 4.1A], there exist $L_K$-invariant subspaces

$$(0) = W_0 \subset W_1 \subset \cdots \subset W_n = V_K$$

with $\dim_K W_i = i$. Suppose $0 \neq v \in V^\lambda$. Let $i$ be greatest with $v \notin W_i$. Let $\bar{v}$ be the image of $v$ in $\overline{W} = W_{i+1}/W_i$. Since $\overline{W}$ is one-dimensional, $\overline{W} = (\overline{W})_\mu = (\overline{W})^\mu$ for some character $\mu$ of $L_K$. It follows that $\lambda = \mu|_L$, hence is a character of $L$.

Now let $L$ be arbitrary. Let $L = S \oplus R$ be the Levi decomposition of $L$, where $S$ is semisimple and $R$ is the radical of $L$ [**4**, §III.9 or **2**, Théorème 1.6.9]. By (a), $\lambda|_S = 0$. By the above, $\lambda|_R$ is a character. Suppose $s \in S$. Then $H = ks \oplus R$ is a solvable Lie algebra, so $\lambda|_H$ is a character. In particular, $\lambda(\alpha s + \beta r) = \alpha\lambda(s) + \beta\lambda(r)$ for all $r \in R$, all $\alpha, \beta \in k$. It follows that $\lambda$ is linear on $L$. Also since $\lambda|_H$ is a character, $\lambda([s, R]) = 0$. Since $[L, L] = S + [S, R] + [R, R]$, this shows that $\lambda$ is a character.

(c) Let $\rho$ be the representation of $L$ corresponding to $V$. Since $\lambda$ is a character, $x \mapsto \rho(x) - \lambda(x)$ is easily seen to be a Lie homomorphism from $L$ to $\mathrm{End}_k V$. Replacing $L$ by its image under this homomorphism, we may assume without loss of generality that $\lambda = 0$. The result now follows from [**1**, Chapter 7, §1, Exercise 5] (due to G. Seligman).

(d) Let notation be as in (b). Since $V^\lambda$ is an $R$ module, $R_K$ acts on $(V^\lambda)_K$ and therefore has, by Lie's Theorem, a common eigenvector $0 \neq v \in (V^\lambda)_K$. Considering $(V^\lambda)_K$ as an $R$ module, $(V^\lambda)_K = ((V^\lambda)_K)^\nu$ where $\nu = \lambda|_R$. Thus $xv = \lambda(x)v$ for all $x \in R$. If $v = \sum v_i \otimes a_i$ where the $a_i \in K$ are linearly independent over $k$ and $v_i \in V^\lambda$, it follows, since $\lambda(x) \in k$, that $xv_i = \lambda(x)v_i$ for each $i$ and all $x \in R$. Since $S$ acts trivially on $v_i$, we thus have $v_i \in V_\lambda$.

Part (a) of the Theorem allows us to remove the restriction of solvability from Corollary 5 of [**7**]. U( ) denotes the universal algebra functor.

COROLLARY. *Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero. Let $B = \bigcap\{\ker\alpha | \alpha \in L^* \text{ and } L_\alpha \neq 0\}$. Then $\mathrm{U}(L)^\lambda \subseteq \mathrm{U}(B)$ for every $\lambda$.*

PROOF. See [**7**, Corollary 4].

REMARK. G. Bergman has suggested, in connection with the proof of the Theorem, that the sum $I$ of all semisimple subalgebras of $L$ should be an ideal of $L$. This does not seem to appear anywhere in the literature, but for $k$ algebraically closed is an immediate consequence of a result of Krempa [**6**, Corollary 1], since $I$ is clearly invariant under all automorphisms of $L$. The following argument applies for any field of characteristic zero. Let $x \in N$, the nilpotent radical of $L$. For $s \in I$ and $\lambda \in k$, $I$ contains

$$\exp(\mathrm{ad}\,\lambda x)s = s + \lambda(\mathrm{ad}\,x)s + \lambda^2(\mathrm{ad}\,x)^2 s + \cdots + \lambda^j(\mathrm{ad}\,x)^j s,$$

where $(\mathrm{ad}\,x)^{j+1} = 0$. Since $k$ is infinite, a Vandermonde determinant argument shows that $(\mathrm{ad}\,x)s \in I$. Thus $[I, N] \subseteq I$. With $S, R$ as above, $R = R_0 + R_1$ where

$[S, R_0] = 0$ and $R_1$ is the sum of nontrivial irreducible $S$ modules ($S$ acting via ad). Then $R_1 = [S, R_1] \subseteq N$, so $[S, R] = [S, R_1] \subseteq [S, N] \subseteq [I, N] \subseteq I$. Thus, $[S, L] = [S, S] + [S, R] \subseteq I$. Since $S$ may be chosen to contain any semisimple subalgebra of $L$, this shows that $I$ is an ideal.

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