GENERALIZATION OF TWO RESULTS
OF THE THEORY OF UNIFORM DISTRIBUTION

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ABSTRACT. For a sequence \( x_1, \ldots, x_N \) of points in \([0,1]\) and a sequence \( p_1, \ldots, p_N \)
\((p_1 + p_2 + \cdots + p_N = 1)\) of nonnegative numbers, define the distribution function
\[
g(x) = x - \sum_{x_k < x} p_k.
\]
Let \( \varphi \) be an increasing function on \([0,1]\) and \( \varphi(0) = 0 \). The main result of the paper
is
\[
F(D_N) \leq \int_0^1 \varphi(|g(x)|) \, dx \leq \varphi(D_N),
\]
where \( D_N \) is the supremum norm of \( g \) on \([0,1]\) and \( F \) is the antiderivative of \( \varphi \) with
\( F(0) = 0 \). This result generalizes and improves an estimate of Niederreiter [1] for the
\( L^2 \) discrepancy of the sequence \( x_1, \ldots, x_N \). Applying the above inequality we also
obtain a new criterion for uniform distribution modulo one.

1. Introduction. Let us recall some definitions and results of the theory of uniform
distribution modulo one.

DEFINITION A. Let \( x_1, x_2, \ldots, x_N \) be a finite sequence in the interval \([0,1]\). The
number
\[
D_N(p) = \left( \int_0^1 \left| x - \frac{1}{N} \sum_{1 \leq k \leq N \atop x_k < x} \right|^p dx \right)^{1/p}, \quad 0 < p \leq \infty,
\]
is called the \( L^p \) discrepancy of the given sequence.

In what follows, we shall write \( D_N \) instead of \( D_N(\infty) \).

H. Niederreiter proved the following theorem using the well-known inequality of
LeVeque.

THEOREM A (NIEDEERREITER [1]). For any sequence \( x_1, x_2, \ldots, x_N \) in \([0,1]\) we have
\[
\frac{1}{\sqrt{12}} D_N^{3/2} \leq D_N^{(2)} \leq D_N.
\]

DEFINITION B. Let \( \sigma = (x_n) \) be an infinite sequence in \([0,1]\). For the infinite
sequence \( \sigma \), the \( L^p \) discrepancy \( D_N^{(p)}(\sigma) \) is defined to be the \( L^p \) discrepancy of the
initial segment formed by the first \( N \) terms of \( \sigma \).

Again, we shall write \( D_N(\sigma) \) instead of \( D_N(\infty)(\sigma) \).

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**Definition C.** An infinite sequence \( \sigma \) in \([0, 1]\) is said to be uniformly distributed (in the sense of Weyl) if
\[
\lim_{N \to \infty} D_N(\sigma) = 0.
\]

The following criterion for uniform distribution is well known.

**Theorem B (Sobol [2, P. 115]).** Suppose \( 0 < p < \infty \). Then an infinite sequence \( \sigma \) in \([0, 1]\) is uniformly distributed if and only if
\[
\lim_{N \to \infty} D_N^{(p)}(\sigma) = 0.
\]

In §2 we generalize and improve Theorem A. In §3 we generalize Theorem B.

2. Generalization of Theorem A. Suppose we are given a finite sequence \( x_1, x_2, \ldots, x_N \) in \([0, 1]\) and a finite sequence \( p_1, p_2, \ldots, p_N \) of nonnegative numbers. We call the numbers \( p_1, p_2, \ldots, p_N \) weights of the numbers \( x_1, x_2, \ldots, x_N \), respectively. Let us define the functions \( h \) and \( g \) on \([0, 1]\) by
\[
(2) \quad h(x) = \sum_{1 \leq k \leq N} p_k \mathbb{1}_{x_k < x}
\]
and
\[
(3) \quad g(x) = x - h(x).
\]
Obviously, the function \( h \) is increasing on \([0, 1]\) and the function \( g \) is piecewise linear on \([0, 1]\).

**Definition 1.** The number
\[
D_N^{(p)} = \left( \int_0^1 |g(x)|^p \, dx \right)^{1/p}, \quad 0 < p \leq \infty,
\]
is said to be the \( L^p \) discrepancy of the sequence \( x_1, x_2, \ldots, x_N \) with respect to the weights \( p_1, p_2, \ldots, p_N \).

Instead of \( D_N^{(\infty)} \) we shall write \( D_N \). Evidently,
\[
(4) \quad D_N = \sup_{0 \leq x \leq 1} |g(x)|.
\]

Comparing Definition A with Definition 1 we see that the \( L^p \) discrepancy of the sequence \( x_1, x_2, \ldots, x_N \) is equal to the \( L^p \) discrepancy of this sequence with respect to the weights \( p_1 = p_2 = \cdots = p_N = 1/N \).

**Theorem 1.** Let \( \varphi \) be an increasing function on \([0, 1]\), \( \varphi(0) = 0 \) and
\[
(5) \quad F(x) = \int_0^x \varphi(t) \, dt.
\]
Then for any sequence \( x_1, x_2, \ldots, x_N \) in \([0, 1]\) and any weights \( p_1, p_2, \ldots, p_N \) with
\[
(6) \quad \sum_{k=1}^N p_k = 1
\]
we have
\[
(7) \quad F(D_N) \leq \int_0^1 \varphi(|g(x)|) \, dx \leq \varphi(D_N),
\]
where the function \( g \) is defined by (3).
Proof. The second inequality in (7) is obvious. It holds true because \( \varphi \) increases on \([0, 1]\). Now, we shall prove the first inequality in (7). Let \( a \) be an arbitrary real number with

\[
0 < a < D_N.
\]

First we shall prove the following inequality.

\[
(9) \quad F(a) \leq \int_0^1 \varphi\left(\left|g(x)\right|\right) \, dx.
\]

It follows from (4) and (8) that there exists a number \( x_0 \in [0, 1] \) such that

\[
\left|g(x_0)\right| > a.
\]

According to the definition of \( g \), the above inequality can be written as

\[
|x_0 - h(x_0)| > a.
\]

Hence, there are two possible cases:

\[
h(x_0) > x_0 + a \quad \text{or} \quad h(x_0) < x_0 - a.
\]

Further, we shall prove (9) in the first case only because it can similarly be proved in the second case as well.

Suppose \( h(x_0) > x_0 + a \). Then from (2) and (6), we have

\[
(10) \quad [x_0, x_0 + a] \subset [0, 1].
\]

Since \( \varphi \) is an increasing function on \([0, 1]\) and \( \varphi(0) = 0 \), it follows that the inequality \( \varphi(|g(x)|) \geq 0 \) holds for every \( x \in [0, 1] \). Hence, we obtain from (10)

\[
(11) \quad \int_0^1 \varphi\left(|g(x)|\right) \, dx \geq \int_{x_0}^{x_0 + a} \varphi\left(|g(x)|\right) \, dx.
\]

Now suppose that \( x \in [x_0, x_0 + a] \). Since the function \( h \) increases on \([0, 1]\), we deduce

\[
g(x) \leq x - h(x_0) < x - x_0 - a < 0.
\]

Therefore,

\[
|g(x)| = -g(x) > x_0 + a - x.
\]

Hence, using (11) and (5) we get

\[
\int_0^1 \varphi\left(|g(x)|\right) \, dx \geq \int_{x_0}^{x_0 + a} \varphi(x_0 + a - x) \, dx = F(a).
\]

Thus, (9) is proved in the first case.

Now it follows from (9) that

\[
(12) \quad \sup_{0 < a < D_N} F(a) \leq \int_0^1 \varphi\left(|g(x)|\right) \, dx.
\]

But the function \( F \) is increasing on \([0, 1]\) because \( \varphi(x) \geq 0 \) for every \( x \in [0, 1] \). Hence, \( F \) is increasing on \([0, D_N]\), too, because \( 0 < D_N \leq 1 \). Therefore,

\[
(13) \quad \sup_{0 < a < D_N} F(a) = F(D_N).
\]

Finally, the first inequality in (7) follows from (12) and (13). Theorem 1 is proved.
Setting \( \phi(x) = x^p \) \((0 < p < \infty)\) in Theorem 1, we immediately obtain

**Corollary 1.** Suppose \(0 < p < \infty\). Then for any sequence \(x_1, x_2, \ldots, x_N\) in \([0,1]\) and any weights \(p_1, p_2, \ldots, p_N\) with (6) we have

\[
\frac{1}{(p + 1)^{1/p}} D_{N+1}^{p + \frac{1}{p}} \leq D_n^{(\phi)} \leq D_N.
\]

**Remark 1.** From (14) we get the following estimate for \(p = 2\),

\[
\frac{1}{\sqrt{3}} D_N^{3/2} \leq D_n^{(2)} \leq D_N.
\]

It is easy to see that (15) improves estimate (1) of Niederreiter.

**Remark 2.** The first inequality in (7) changes into an equality if \(x_1 = x_2 = \cdots = x_N = 0\).

### 3. Generalization of Theorem B

Suppose we are given two infinite triangular matrices \(X = (x_{nk}^{(n)})\) and \(P = (p_{nk}^{(n)})\) with \(0 \leq x_{nk}^{(n)} \leq 1\) and \(p_{nk}^{(n)} > 0\) \((n = 1, 2, \ldots; k = 1, \ldots, n)\). We call the matrix \(P\) a weight matrix of the matrix \(X\).

**Definition 2.** Suppose \(0 < p < \infty\). The \(L^p\) discrepancy \(D_n^{(p)}(X, P)\) is defined to be the \(L^p\) discrepancy of the sequence \(x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}\) with respect to the weights \(p_1^{(n)}, p_2^{(n)}, \ldots, p_n^{(n)}\), i.e.

\[
D_n^{(p)}(X, P) = \left( \int_0^1 \left( \sum_{1 \leq k \leq n} p_{nk}^{(n)} g_{nk}(x) \right)^p dx \right)^{1/p},
\]

where

\[
g_{nk}(x) = x - \sum_{1 \leq k \leq n} p_{nk}^{(n)} x_{nk}^{(n)} < x.
\]

**Definition 3 (see [3]).** The matrix \(X\) is said to be uniformly distributed with respect to the weight matrix \(P\) if

\[
\lim_{n \to \infty} D_n(X, P) = 0.
\]

**Definition 4.** Let \(\phi\) be a function defined on \([0,1]\). We call \(\phi\) a basic function if it satisfies the following three conditions:

(i) \(\phi\) is increasing on \([0,1]\),

(ii) \(\lim_{x \to 0^+} \phi(x) = 0\),

(iii) \(\phi(x) = 0\) if and only if \(x = 0\).

**Lemma 1.** Let \(\phi\) be a basic function. Then the function \(F\) defined by (5) is a basic function as well.

**Proof.** From (5), (i) and (iii), we deduce

\[
F(x_2) - F(x_1) > \frac{x_2 - x_1}{2} \phi \left( \frac{x_1 + x_2}{2} \right) > 0
\]

for all \(x_1\) and \(x_2\) with \(0 \leq x_1 < x_2 \leq 1\). Therefore, \(F\) is strictly increasing on \([0,1]\). Hence, for every \(x \in (0,1]\) we have

\[
0 = F(0) < F(x) \leq x \phi(x).
\]

Passing to the limit as \(x \to 0^+\) in this inequality, we get \(\lim_{x \to 0^+} F(x) = 0\).
Lemma 2. Let \( \varphi \) be a basic function. Then the matrix is uniformly distributed with respect to the weight matrix if and only if

\[
\lim_{n \to \infty} \varphi(D_n(X, P)) = 0.
\]

Proof. The necessity follows immediately from (ii) and Definition 3. Now suppose that (19) holds, but (18) does not hold. Then there exists a positive number \( \varepsilon_0 \) such that the inequality

\[
D_n(X, P) \geq \varepsilon_0
\]

holds for infinitely many values of \( n \). It follows from (20) and (i) that

\[
\varphi(D_n(X, P)) \geq \varphi(\varepsilon_0).
\]

From (19), (21) and (iii), we deduce

\[
0 = \lim_{n \to \infty} \varphi(D_n(X, P)) \geq \varphi(\varepsilon_0) > 0,
\]

which is a contradiction. Therefore, if (19) holds then (18) holds, too, i.e. \( X \) is uniformly distributed with respect to \( P \).

The following criterion for uniform distribution is a generalization of Theorem B.

Theorem 2. Let \( \varphi \) be a basic function and let \( P \) be a weight matrix with

\[
\sum_{k=1}^{n} p_k^{(n)} = 1 \quad (n = 1, 2, \ldots).
\]

Then a matrix \( X \) is uniformly distributed with respect to the weight matrix \( P \) if and only if

\[
\lim_{n \to \infty} \int_{0}^{1} \varphi(|g_n(x)|) \, dx = 0,
\]

where \( g_n(x) \) is defined by (17).

Proof. By Theorem 1 we have

\[
F(D_n(X, P)) \leq \int_{0}^{1} \varphi(|g_n(x)|) \, dx \leq \varphi(D_n(X, P)).
\]

Since \( \varphi \) is a basic function, it follows from Lemma 1 that \( F \) is a basic function, too. Now, the assertion follows from (23) and Lemma 2.

Setting \( \varphi(x) = x^p \) \( (0 < p < \infty) \) in Theorem 2, we immediately obtain

Corollary 2. Let \( 0 < p < \infty \) and \( P \) be a weight matrix with (22). Then a matrix \( X \) is uniformly distributed with respect to the weight matrix \( P \) if and only if

\[
\lim_{n \to \infty} D_n^{(p)}(X, P) = 0.
\]

Remark 3. It is easy to see that Corollary 2 is a generalization of Theorem B. Indeed, let \( a = (x_n) \) be an infinite sequence in \([0,1]\). Applying Corollary 2 for the matrices \( X = (x_k^{(n)}) \) and \( P = (p_k^{(n)}) \) with \( x_k^{(n)} = x_k \) and \( p_k^{(n)} = 1/n \) \( (n = 1, 2, \ldots; k = 1, \ldots, n) \) we get Theorem B.
4. **Final remark.** Theorem 2 shows that as a measure of the distribution of a matrix $X$ with respect to a weight matrix $P$, alongside with the $L^p$ discrepancy $D_n^{(p)}(X, P)$, one may use the $\varphi$-discrepancy

$$D_n^{(\varphi)}(X, P) = \int_0^1 \varphi(|g_n(x)|) \, dx,$$

where $\varphi$ is a basic function and $g_n(x)$ is defined by (17).

Similarly, as a measure of the distribution of a sequence $x_1, x_2, \ldots, x_N$ in $[0, 1]$ with respect to the weights $p_1, p_2, \ldots, p_N$, alongside with the $L^p$ discrepancy $D_N^{(p)}$, one can use the $\varphi$-discrepancy

$$D_N^{(\varphi)} = \int_0^1 \varphi(|g(x)|) \, dx,$$

where $\varphi$ is a basic function, too, and $g(x)$ is defined by (3).

**References**