CONSTRUCTION OF A DIFFERENTIAL EQUATION $y'' + Ay = 0$
WITH SOLUTIONS HAVING THE PRESCRIBED ZEROS

LI-CHIEN SHEN

ABSTRACT. We show that an entire function $A(z)$ can be constructed so that
the differential equation $y'' + Ay = 0$ has two linearly independent solutions
having the prescribed points as their only zeros.

The main purpose of this brief note is to prove

THEOREM 1. Let $\{a_n\}$ and $\{b_n\}$ be two given sequences with no finite limit
point. If the two sequences have no points in common, that is $a_n \neq b_m$ for any $n$
and $m$ ($n, m = 1, 2, 3, \ldots$), then there exists an entire function $A(z)$ such that
the differential equation $y'' + Ay = 0$ has two linearly independent solutions $w_1$ and $w_2$
whose only zeros are $\{a_n\}$ and $\{b_n\}$, respectively.

1. Preliminaries. Consider the differential equation (D.E.)
$$y'' + A(z)y = 0,$$
where $A$ is entire. It is well known that
(i) all the solution of (1.1) are entire;
(ii) if $w_1$ and $w_2$ are two linearly independent solutions, then the Wronskian
$W(z; w_1, w_2) = w_1 w_2' - w_1' w_2$ is constant, with no loss of generality, we assume that
this constant is 1;
(iii) if $z_0$ is a zero of a solution, then its multiplicity is always equal to 1.

We say that an entire function $f$ has the BL property if, for each one of its
zeros, say $a$, we have either $f'(a) = 1$ or $f'(a) = -1$.

Set
$$f(z) = w_1(z) \cdot w_2(z).$$
It is easy to derive that $f$ satisfies the D.E.
$$-4Af^2 = 1 - (f')^2 + 2ff''.$$
From (1.3) we readily conclude

LEMMA 1.1. Let $w_1$ and $w_2$ be two linearly independent solutions of (1.1).
Then $f = w_1 w_2$ has the BL property.

It is an elementary exercise to show that if an entire function $f$ has the BL
property, then the function $A$ defined by
$$-4A(z) = (1/f^2) - (f'/f)^2 + 2(f''/f)$$
is entire. With this fact in mind, we now establish

**Lemma 1.2.** Let \( f \) be an entire function with the BL property. Then \( f = w_1 w_2 \), where \( w_1 \) and \( w_2 \) are some linearly independent solutions of D.E. with \( A \) defined by (1.4).

**Proof.** Let \( z_0 \) be a point such that \( f(z_0) \neq 0 \). Then we can choose a disk \( D = \{ z : |z - z_0| < r_0 \} \) with the property that \( f(z) \neq 0 \) for all \( z \in D \). In \( D \), we define (by choosing a branch)

\[
(1.5) \quad w_1(z) = (f)^{1/2} \exp \left( -\frac{1}{2} \int_{z_0}^{z} \frac{1}{f(s)} \, ds \right),
\]

and

\[
(1.6) \quad w_2(z) = (f)^{1/2} \exp \left( \frac{1}{2} \int_{z_0}^{z} \frac{1}{f(s)} \, ds \right).
\]

Although it is not clear at the outset that \( w_1 \) and \( w_2 \) can be analytically continued uniquely to the entire complex plane, by a straightforward substitution, however, it can be easily shown that \( w_1 \) and \( w_2 \) both satisfy the D.E. (1.1) with \( A \) defined by (1.4). Thus, from (i), we conclude that \( w_1 \) and \( w_2 \) are both entire and \( f = w_1 w_2 \).

From Lemmas 1.1 and 1.2, we conclude

**Corollary 1.3.** An entire function \( f \) has the BL property iff \( f \) is a product of two linearly independent solutions of D.E. (1.1).

2. **Proof of Theorem 1.** Choose an entire function \( g(z) \) so that its only zeros are \( \{a_n\} \cup \{b_n\} \) and the multiplicity of each zero is one. We also choose an entire function \( h \) such that

\[
(2.1) \quad \exp(h(z)) = \begin{cases} 
-1/g'(a_n) & \text{if } z = a_n, \\
1/g'(b_n) & \text{if } z = b_n.
\end{cases}
\]

The algorithms to construct \( g \) and \( h \) are well known; however, such \( g \) and \( h \) are not unique \([1, \text{pp. } 295 \text{ and } 298]\).

Define

\[
(2.2) \quad f = g \exp(h).
\]

Then, from (2.1), \( f \) has the BL property and

\[
(2.3) \quad f'(z) = \begin{cases}
-1 & \text{if } z = a_n, \\
1 & \text{if } z = b_n.
\end{cases}
\]

Thus, from Lemma 1.2, there exists an entire function \( A \) and \( f = w_1 w_2 \), where \( w_1 \) and \( w_2 \) are two linearly independent solutions of the D.E. \( y'' + A(z)y = 0 \) defined by (1.5) and (1.6) respectively. We now show that \( w_1 \) and \( w_2 \) have the desired property.

Let

\[
(2.4) \quad F(z) = \exp \left( \int_{z_0}^{z} 1/f(s) \, ds \right).
\]
From (1.5), $F(z) = f/w_1^2$. Therefore, $F(z)$ is meromorphic. From (2.3), we see that in a small neighborhood $U$ of $z = a_n$

\begin{equation}
1/f(z) = -1/(z - a_n) + H_n(z),
\end{equation}

where $H_n$ is holomorphic in $U$. From (2.4) and (2.5),

\begin{equation}
F'/F = 1/f = -1/(z - a_n) + H_n(z) \quad (z \in U).
\end{equation}

Therefore, (2.6) implies that $F$ has a simple pole at $z = a_n$. A similar argument shows that $F$ has a simple zero at $z = b_n$. Since the only zeros of $f$ are $\{a_n\} \cup \{b_n\}$, hence $\{a_n\}$ and $\{b_n\}$ are the only poles and zeros of $F$, respectively. This immediately implies that $F'' (= F'/f)$ has no zeros and has double poles at $\{a_n\}$. Thus $w_1 = 1/(F'')^{1/2}$ is an entire function whose zeros are precisely $\{a_n\}$. Since $w_2 = w_1 F$, the only zeros of $w_2$ are $\{b_n\}$. This completes the proof.

REFERENCES


DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NEW YORK 13210

Current address: Department of Mathematics, University of Florida, Gainesville, Florida 32611