ON A 2-DIMENSIONAL EINSTEIN KAEPHER SUBMANIFOLD OF A COMPLEX SPACE FORM

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Abstract. In this paper we consider when a Kaehler submanifold of a complex space form is Einstein with respect to the induced metric. Then we shall show that
(1) a 2-dimensional complete Kaehler submanifold $M$ of a 4-dimensional complex projective space $P^4(C)$ is Einstein if and only if $M$ is holomorphically isometric to $P^2(C)$ which is totally geodesic in $P^4(C)$ or a hyperquadric $Q^2(C)$ in $P^3(C)$ which is totally geodesic in $P^4(C)$, and that
(2) if $M$ is a 2-dimensional Einstein Kaehler submanifold of a 4-dimensional complex space form $M^4(\tilde{c})$ of nonpositive constant holomorphic sectional curvature $\tilde{c}$, then $M$ is totally geodesic.

Let $M^n$ be an $n$-dimensional Kaehler submanifold of an $(n + p)$-dimensional complex space form $\tilde{M}^{n+p}(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$. Let $P^{n+p}(C)$ be an $(n + p)$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1, and let $C^{n+p}$ (or $D^{n+p}$) be a complex $(n + p)$-dimensional Euclidean space with the usual Hermitian metric (resp. the $(n + p)$-dimensional open unit ball in $C^{n+p}$ with the natural complex structure and the Bergman metric of constant holomorphic sectional curvature $-1$). Any $m$-dimensional complex space form is (after multiplying the metric by a suitable constant) locally complex analytically isometric to $P^m(C)$, $C^m$ or $D^m$, according as the holomorphic sectional curvature is positive, zero or negative. We consider the following question:

When is $M$ Einstein with respect to the induced metric?

The answer is known in the following cases:

(1) $p = 1$ (Smyth [8]).

A complete hypersurface $M$ of $P^{n+1}(C)$ is Einstein if and only if $M$ is holomorphically isometric to $P^n(C)$ or a complex hyperquadric $Q^n(C)$.

A complete and simply connected hypersurface $M$ of $C^{n+1}$ (resp. $D^{n+1}$) is Einstein if and only if $M$ is holomorphically isometric to $C^n$ (resp. $D^n$).

(2) A complete intersection submanifold (Hano [3]).

A complete intersection submanifold $M^n$ of $P^{n+p}(C)$ is Einstein if and only if $M$ is $P^n(C)$ which is totally geodesic in $P^{n+p}(C)$ or a hyperquadric $Q^n(C)$ in $P^{n+1}(C)$ which is totally geodesic in $P^{n+p}(C)$.
In this paper we consider the case of $n = 2$. We show

**Theorem A.** A complete Kaehler submanifold $M^2$ of $P^4(C)$ is Einstein if and only if $M$ is holomorphically isometric to $P^2(C)$ which is totally geodesic in $P^4(C)$ or a hyperquadric $Q^2(C)$ in $P^3(C)$ which is totally geodesic in $P^4(C)$.

Since for each $n > 1$ there exists a Kaehler imbedding $f: P^n_{1/2}(C) \rightarrow P^{n+(n(n+1))/2}(C)$ (O'Neill [7]), the codimension of Theorem A is best possible, where $1/2$ of $P^n_{1/2}(C)$ denotes constant holomorphic sectional curvature.

**Theorem B.** Let $M^2$ be an Einstein Kaehler submanifold immersed in $\tilde{M}^4(\tilde{\xi})$, $\tilde{\xi} \leq 0$. Then $M$ is totally geodesic.

**1. Kaehler submanifolds.** As in the introduction, let $M^n$ be a Kaehler submanifold of $\tilde{M}^{n+p}(\tilde{\xi})$. And let $\tilde{f}$ be its Kaehler immersion (i.e., holomorphically isometric immersion). Let $J_0$ (resp. $J$) be the complex structure of $M$ (resp. $\tilde{M}$). In order to simplify the presentation we identify, for each $x \in M$, the tangent space $T_x(M)$ with $f_*(T_x(M)) \subset T_{f(x)}(\tilde{M})$ by means of $f_*$. The normal space $T^\perp_x(M)$ is the subspace of $T_{f(x)}(\tilde{M})$ consisting of all $X \in T_{f(x)}(\tilde{M})$ which are orthogonal, with respect to $g$, to the subspace $f_*(T_x(M))$. Since $f^*g = g_0$ and $J = f_0 \circ J_0$, where $J_0$ is the almost complex structure of $M$, the structures $g_0$ and $J_0$ on $T_x(M)$ are respectively identified with the restrictions of the structures $g$ and $J$ to the subspace $f_*(T_x(M))$. With this identification in mind we drop the subscript $0$ on $g_0$ and $J_0$. Let $\nabla$ (resp. $\tilde{\nabla}$) denote the covariant differentiation in $M$ (resp. $\tilde{M}$), and let $\nabla^\perp$ denote covariant differentiation in the normal bundle.

With each $\xi \in T^\perp_x(M)$ is associated a linear transformation of $T_x(M)$ in the following way. Extend $\xi$ to a normal vector field defined in a neighborhood of $x$ and define $-A_\xi X$ to be the tangential component of $\tilde{\nabla}_X \xi$ for $X \in T_x(M)$. $A_\xi X$ depends only on $\xi$ at $x$ and $X$. Given an orthonormal basis $\xi_1, \ldots, \xi_p, \xi_{1^*}, \ldots, \xi_{p^*}$ of $T^\perp_x(M)$, we write $A_\alpha = A_{\xi_\alpha}$ and call the $A_\alpha$'s the second fundamental forms associated with $\xi_1, \ldots, \xi_p, \xi_{1^*}, \ldots, \xi_{p^*}$, are now orthonormal normal vector field in a neighborhood $U$ of $x$, they determine normal connection forms $s_{\alpha\beta}, t_{\alpha\beta}$ in $U$ by

$$\begin{align*}
\nabla^\perp_X \xi_\alpha &= \sum_\beta s_{\alpha\beta}(X) \xi_\beta + \sum_\beta t_{\beta\alpha}(X) \xi_{\beta^*}, \\
s_{\alpha\beta} + s_{\beta\alpha} &= 0, t_{\alpha\beta} - t_{\beta\alpha} = 0
\end{align*}$$

for $X \in T_x(M)$. Let $S$ and $\rho$ be the Ricci tensor and the scalar curvature for $M$, respectively. Then we have the following relationship (in this paper $\alpha, \beta, \gamma$ run from 1 to $p$ and $\lambda, \mu$ run from $1^*, \ldots, p^*$, except when noted) [5]:

$$\begin{align*}
\tilde{\nabla}_X Y &= \nabla_X Y + \sigma(X, Y), \\
\sigma(X, Y) &= \sum g(A_\alpha X, Y) \xi_\alpha, \\
g(A_\alpha X, Y) &= g(A_\alpha Y, X), \quad A_{\alpha^*} = JA_\alpha, \quad A_{\alpha}J = -A_{\alpha}J, \\
(\nabla_X A_\alpha)Y - \sum_\beta s_{\beta\alpha}(X) A_\beta Y - \sum_\beta t_{\beta\alpha}(X) JA_\beta Y &= (\nabla_Y A_\alpha)X - \sum_\beta s_{\beta\alpha}(Y) A_\beta X - \sum_\beta t_{\beta\alpha}(Y) JA_\beta X - \text{Codazzi equation},
\end{align*}$$
\begin{align}
S &= \frac{n+1}{2} \tilde{c} I - 2 \sum_{\alpha} A_{\alpha}^2, \\
\rho &= n(n + 1) \tilde{c} - \|\sigma\|^2,
\end{align}

where \( X, Y \) are tangent to \( M \), \( \sigma \) is also called the second fundamental form of \( f \), and \( I, \|\sigma\| \) denote the identity transformation of \( T_x(M) \), the length of \( \sigma \), so that

\[ \|\sigma\|^2 = 2 \sum \text{trace} \, A_{\alpha}. \]

2. Proposition. Let \( M^n \) be a Kaehler submanifold of \( \tilde{M}^{n+p}(\tilde{\tilde{c}}) \).

**Definition.** For \( x \in M \), the first normal space, \( N_1(x) \), is the orthogonal complement in \( T_x(M) \) of the set \( N_0(x) = \{ \xi \in T_x(M) | A_\xi = 0 \} \).

**Proposition 1.** Let \( M^2 \) be a 2-dimensional Einstein Kaehler submanifold of \( \tilde{M}^{2+2}(\tilde{\tilde{c}}) \). Then dim \( N_1(x) \neq 2 \).

**Proof.** We assume that dim \( N_1(x) = 2 \). Then dim \( N_1(y) \geq 2 \) in a neighborhood \( U_0 \) of \( x \). In terms of codimension \( M = 2 \) we have dim \( N_1(y) = 2 \) in \( U_0 \). Choose local orthonormal vector fields \( \xi_1, \xi_2, J\xi_1, J\xi_2 \) so that a symmetric \((4,4)\)-matrix (trace \( A_\lambda A_\mu \)) can be diagonalized.

Let \( \Omega = \{ e_1, e_2, e_1^{*} = Je_1, e_2^{*} = Je_2 \} \) be an orthonormal basis for \( T_x(M) \) with respect to which \( A_1 \) is diagonal of the form

\[ A_1 = \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}. \]

By means of

\[ g(A_2 e_1, e_2^{*}) = g(A_2 e_2, e_1^{*}), \]
\[ g(A_2 e_1^{*}, e_2^{*}) = -g(A_2 e_1, e_2), \quad a, b = 1, 2, \]

\( A_2 \) is represented by the matrix

\[ A_2 = \begin{pmatrix} a_{11} & a_{12} & a_{11}^{*} & a_{12}^{*} \\ a_{12} & a_{22} & a_{12}^{*} & a_{22}^{*} \\ a_{11}^{*} & a_{12}^{*} & -a_{11} & -a_{12} \\ a_{12}^{*} & a_{22}^{*} & -a_{12} & -a_{22} \end{pmatrix}. \]

Since \( M^2 \) is Einstein, i.e., \( S = (\rho/(2 \text{dim } M)) I \), we have

\[ A_1^2 + A_2^2 = \frac{1}{2}\|\sigma\|^2 I. \]

From trace \( A_1 A_2 = \text{trace} \, A_1 A_2^{\ast} = 0 \) we have

\[ a_{11} \alpha + a_{22} \beta = 0, \]
\[ a_{11}^{\ast} \alpha + a_{22}^{\ast} \beta = 0. \]

**Case 1.** \( a_{11} = a_{22} = a_{11}^{\ast} = a_{22}^{\ast} = 0. \) Then \( A_2^2 = (a_{12}^2 + a_{12}^{\ast 2}) I \). Thus from (2.1) we obtain \( \alpha^2 = \beta^2 \), i.e., \( A_1^2 = \alpha^2 I \). Then

\[ A_1 \circ A_{\xi_1 + \xi_2} = A_{\xi_1 + \xi_2} \circ A_1, \]
Hence $A_1$, $A_2^2$, and $A_1^2 + A_2^2$ can be simultaneously diagonalized. We may assume that from the beginning we choose such a basis $\Omega$. Then we obtain

$$A_1^2 + A_2^2 = A_1 A_2 + A_2 A_1$$

Therefore

$$a_{1\alpha} + a_{1\beta} = 0, \quad a_{1\alpha} - a_{1\beta} = 0, \quad a_{2\alpha} + a_{2\beta} = 0, \quad a_{2\alpha} - a_{2\beta} = 0.$$ 

Since we may assume that $(\alpha \beta) \neq 0$, we obtain $a_{1\alpha} = a_{1\beta} = 0$. This contradicts the assumption that $k = 2$.

**Case 2.** $(a_{11} a_{22} a_{11*} a_{22*}) \neq 0$. Since $A_2^2$ is diagonal, we obtain

$$(a_{11} + a_{22}) a_{1\alpha} + (a_{11} + a_{22}) a_{1\beta} = 0,$$

$$(a_{22} - a_{11}) a_{2\alpha} + (a_{11} - a_{22}) a_{2\beta} = 0.$$ 

If $(a_{1\alpha} a_{2\alpha}) \neq 0$, i.e., $a_{11}^2 + a_{11*}^2 = a_{22}^2 + a_{22*}^2$, then we can put $A_2^2 = a^2 I$, where

$$a^2 = a_{11}^2 + a_{11*}^2 + a_{12}^2 + a_{12*}^2 = a_{22}^2 + a_{12}^2 + a_{22*}^2 + a_{12*}^2.$$ 

Combining this fact with (2.1), we have $\alpha^2 = \beta^2$, i.e., $A_1^2 = a^2 I$. Then we have

$$A_1^2 + A_2^2 = A_1 + A_2^2,$$

$$A_1 + A_2^2 = A_1 + A_2^2.$$ 

Thus by the same argument as Case 1 we know that the case of $(a_{1\alpha} a_{1\beta}) \neq 0$ cannot occur. Thus we may assume that $a_{1\alpha} = a_{1\beta} = 0$. Then from (2.1) we have

$$\alpha^2 + a_{11}^2 + a_{11*}^2 = \frac{1}{2} ||\sigma||^2,$$

$$\beta^2 + a_{22}^2 + a_{22*}^2 = \frac{1}{2} ||\sigma||^2.$$ 

Multiplying (2.4) (resp. (2.5)), by $\alpha^2$ (resp. $\beta^2$) we have

$$\alpha^4 + a_{11}\alpha^2 + a_{11*}\alpha^2 = \frac{1}{8} \alpha^2 ||\sigma||^2,$$

$$\beta^4 + a_{22}\beta^2 + a_{22*}\beta^2 = \frac{1}{8} \beta^2 ||\sigma||^2.$$
Using (2.2) and (2.3), we get from (2.6) that
\[(2.6)' \quad \alpha^2 + a_{22}^2 \beta^2 + a_{22*}^2 \beta^2 = \frac{1}{2} \alpha^2 \|\sigma\|^2.\]
Subtracting (2.7) from (2.6)', we obtain
\[(\alpha^2 - \beta^2)(\alpha^2 + \beta^2 - \frac{1}{2} \|\sigma\|^2) = 0.\]
If \(\alpha^2 \neq \beta^2\), then from (2.4) and (2.5) we have
\[A_2^2 = \begin{pmatrix}
\beta^2 \\
\alpha^2 \\
\beta^2 \\
\alpha^2
\end{pmatrix}.\]
Hence we know that \(\det A_2^2 = \det A_2^2\). Since we have
\[\det A_1 = \alpha^2 \beta^2, \quad \det A_2 = a_{11}^2 a_{22}^2 + a_{11}^2 a_{22*}^2 + a_{11}^2 a_{22}^2 + a_{11}^2 a_{22*}^2,\]
we obtain
\[(2.8) \quad \alpha^2 \beta^2 = a_{11}^2 a_{22}^2 + a_{11}^2 a_{22*}^2 + a_{11}^2 a_{22}^2 + a_{11}^2 a_{22*}^2.\]
On the other hand, from (2.2) and (2.3) we have
\[(2.9) \quad a_{11} a_{22*} - a_{11} a_{22} = 0,\]
since we may assume that \((\alpha \beta) \neq 0\). Combining (2.8) and (2.9), we obtain
\[(2.10) \quad \alpha^2 \beta^2 = (a_{11} a_{22} + a_{11} a_{22*})^2.\]
If \(\alpha \beta = a_{11} a_{22} + a_{11} a_{22*}\), then multiplying (2.2) by \(\alpha\) and using (2.9), we have
\[a_{11}(a_{22}^2 + a_{22*}^2) = 0; \quad \text{i.e.,} \quad a_{11} \alpha^2 = 0.\]
Similarly, we have
\[a_{22}(a_{11}^2 + a_{11*}^2) = 0; \quad \text{i.e.,} \quad a_{22} \beta^2 = 0,\]
\[a_{11*}(a_{22}^2 + a_{22*}^2) = 0; \quad \text{i.e.,} \quad a_{11*} \alpha^2 = 0,\]
\[a_{22*}(a_{11}^2 + a_{11*}^2) = 0; \quad \text{i.e.,} \quad a_{22*} \beta^2 = 0.\]
Hence, we obtain
\[\begin{pmatrix}
\alpha^2 \\
\beta^2 \\
\alpha^2 \\
\beta^2
\end{pmatrix} = \begin{pmatrix}
a_{11} \\
a_{22} \\
a_{11*} \\
a_{22*}
\end{pmatrix} = 0.\]
Since \((a_{11} a_{22} a_{11*} a_{22*}) \neq 0\), we have \(\alpha \beta = 0\). Assume that \(\alpha \beta \neq 0\). Then from (2.10) we have \(\alpha \beta + a_{11} a_{22} + a_{11*} a_{22*} = 0\) and
\[\sigma(e_1, e_1) = \alpha \xi_1 + a_{11} \xi_2 - a_{11*} \xi_2*,\]
\[\sigma(e_2, e_2) = \beta \xi_1 + a_{22} \xi_2 - a_{22*} \xi_2*.\]
Hence, we obtain
\[g(\sigma(e_1, e_1), \sigma(e_2, e_2)) = g(\sigma(e_1, e_1), \sigma(e_2, e_2*)) = 0\]
and \(\sigma(e_1, e_2) = 0\). On the other hand, from (2.1) we have
\[(2.11) \quad \|\sigma(E_a, E_2)\|^2 + \|\sigma(E_1, E_2)\|^2 = \frac{1}{2} \|\sigma\|^2, \quad \alpha = 1, 2,\]
(2.12) \[ g(\sigma(E_1, E_1), \sigma(E_1, E_2)) + g(\sigma(E_2, E_2), \sigma(E_1, E_2)) = 0, \]
(2.13) \[ g(\sigma(E_1, E_1), \sigma(E_1, E_2^*)) - g(\sigma(E_2, E_2), \sigma(E_1, E_2^*)) = 0, \]
where \( \{E_1, E_2, E_1^*, E_2^*\} \) is a suitable local orthonormal frame around \( x \) such that \( E_j = e_j \) at \( x, j = 1, 2, 1^*, 2^* \). Then from (2.11) we have
\[ g(\nabla_X^\perp (\sigma(E_1, E_1)), \sigma(E_1, E_2)) + g(\nabla_X^\perp (\sigma(E_1, E_2)), \sigma(E_1, E_2)) = 0, \]
i.e.,
\[ g(\nabla_X^\perp (\sigma(E_1, E_2)), \sigma(e_1, e_2)) = 0 \text{ at } x. \]
Since \( \sigma(e_1, e_2) = 0 \) at \( x \), we have [5]
\[ (\nabla_X^\perp \sigma)(e_1, e_2) = \nabla_X^\perp (\sigma(E_1, E_2)) - \sigma(\nabla_X E_1, E_1) - \sigma(E_1, \nabla_X E_1) \]
\[ = \nabla_X^\perp (\sigma(E_1, E_2)) - 2g(\nabla_X E_1, E_2^*) \sigma(e_1, e_2^*), \]
where \( \nabla' \) denotes covariant differentiation with respect to the connection in (tangent bundle) \( \oplus \) (normal bundle). Hence, we obtain
(2.14) \[ g((\nabla_X^\perp \sigma)(e_1, e_2), \sigma(e_1, e_2)) = 0. \]
From (2.12) and \( \sigma(e_1, e_2) = 0 \) at \( x \) we have
\[ g(\nabla_X^\perp (\sigma(E_1, E_2)), \sigma(e_1, e_1)) + g(\nabla_X^\perp (\sigma(E_1, E_2)), \sigma(e_2, e_2)) = 0. \]
Using (2.11) and
(2.15) \[ (\nabla_X^\perp \sigma)(e_1, e_2) = \nabla_X^\perp (\sigma(E_1, E_2)) - g(\nabla_X E_1, E_2^*) \sigma(e_2, e_2) \]
\[ - g(\nabla_X E_1, E_2^*) \sigma(e_2, e_2) - g(\nabla_X E_2, E_1) \sigma(e_1, e_1) \]
\[ - g(\nabla_X E_2, E_1^*) \sigma(e_1, e_1^*), \]
we obtain
\[ g((\nabla_X^\perp \sigma)(e_1, e_2), \sigma(e_1, e_1)) + g((\nabla_X^\perp \sigma)(e_1, e_2), \sigma(e_2, e_2)) + g(\nabla_X E_1, E_2) \|\sigma(e_1, e_2)\|^2 = 0, \]
i.e.,
\[ g((\nabla_X^\perp \sigma)(e_1, e_2), \sigma(e_1, e_1)) + g((\nabla_X^\perp \sigma)(e_1, e_2), \sigma(e_2, e_2)) = 0. \]
Since from (2.14) we obtain
\[ g((\nabla_X^\perp \sigma)(e_1, e_2), \sigma(e_1, e_1)) = g((\nabla_X^\perp \sigma)(e_1, e_1), \sigma(e_1, e_1)) = 0, \]
we have
\[ g((\nabla_X^\perp \sigma)(e_1, e_1), \sigma(e_1, e_2)) = g((\nabla_X^\perp \sigma)(e_1, e_1), \sigma(e_2, e_2)) = 0. \]
Similarly, from (2.13) we have
\[ g((\nabla_X^\perp \sigma)(e_1, e_1), \sigma(e_1, e_2^*)) = 0. \]
Hence,
\[ g((\nabla_X^\perp \sigma)(e_1, e_1), \sigma(e_1, e_1^*)) = g((\nabla_X^\perp \sigma)(e_1, e_1), \sigma(e_2, e_2^*)) = 0. \]
Thus
\[(2.16) \quad \left( \nabla' e_2 \sigma \right)(e_1, e_1) = 0 \quad \text{at } x.\]
Similarly, we obtain
\[(2.17) \quad \left( \nabla' e_2 \sigma \right)(e_2, e_2) = 0 \quad \text{at } x.\]
Since \(\alpha \beta (\alpha^2 - \beta^2) \neq 0\), \(\alpha \beta (\alpha^2 - \beta^2) \neq 0\) in a neighborhood \(U\) of \(x\). Using the minimal polynomial of \(A_1\), we can show that \(\{ X | A_1 X = \gamma X \}\) is differentiable (e.g. [4]), where \(\gamma\) is the eigenvalue of \(A_1\). Thus we may assume that \(A_1\) is diagonalized with respect to \(\{ E_1, E_2, E_1, E_2, \ldots \}\) in \(U\). Then we have
\[
g(\sigma(E_1, E_1), \sigma(E_2, E_2)) = 0.
\]
Thus
\[
g\left( \nabla' \left( (\sigma(E_1, E_1)), \sigma(E_2, E_2) \right) + g\left( \sigma(E_1, E_1), \nabla' \left( \sigma(E_2, E_2) \right) \right) \right) = 0.
\]
Hence,
\[
g\left( \left( \nabla' e_2 \sigma \right)(e_1, e_1), \sigma(e_2, e_2) \right) = 0,
g\left( \left( \nabla' e_2 \sigma \right)(e_2, e_2), \sigma(e_1, e_1) \right) = 0 \quad \text{at } x.
\]
Therefore we know that \(\nabla' \sigma = 0\) at \(x\). On the other hand, from the equation [6]
\[
\frac{1}{2} \Delta \|\sigma\|^2 = \|\nabla' \sigma\|^2 - 8 \text{ trace } \left( \sum A_\alpha^2 \right)^2 - \sum (\text{trace } A_\lambda A_\mu)^2 + 2 \|\sigma\|^2,
\]
we obtain \(\|\sigma\|^2 = 0\) [5]. This is a contradiction. We know that the case \(\alpha \beta \neq 0\) and \(\alpha \beta + a_{11}a_{22} + a_{11}a_{22} = 0\) cannot occur. By the similar argument with the above case we know that both cases \(\alpha \beta = 0\) and \(\alpha^2 = \beta^2\) cannot occur. This completes the proof.

3. Proofs of Theorems A and B. From Proposition 1 we know that \(k \leq 1\).

**PROPOSITION 2.** \(N_i(x)\) has constant dimension on \(M^2\).

**PROOF.** If the second fundamental form \(\sigma(X, Y) = 0\) for all \(x \in M^2\), then \(N_i(x)\) has constant dimension 0, and the proofs are complete.

Suppose that \(\sigma(X, Y) \neq 0\) at \(x_0 \in M^2\). Let \(U\) be a neighborhood of \(x_0\) on which \(\sigma(X, Y) \neq 0\). Then \(N_i(u), u \in U\), has constant dimension 1 on \(U\). Since \(M^2\) is Einstein,
\[
A_1 = \begin{pmatrix} \alpha & \alpha \\ \alpha & -\alpha \end{pmatrix},
\]
where \(\alpha = (1/2\sqrt{2})\|\sigma\|\). This implies \(\alpha\) is constant on \(U\). Consider the set \(S\) defined by
\[
S = \{ x \in M^2 | \alpha(x) = \alpha(x_0) \}.
\]
Since \(\alpha\) is continuous on \(M^2\), we know \(S\) is closed. However, the above argument
implies \( S \) is open. Since \( x_0 \in S \), we know \( S \neq \emptyset \); so the connectedness of \( M^2 \)
implies \( S = M^2 \). Hence, \( \alpha = \alpha(x_0) \) on \( M^2 \), and \( N_1(x) \) has constant dimension 1 on \( M^2 \).

In the case where \( N_1(x) \) has constant dimension 0, \( M^2 \) is totally geodesic. To complete the proofs of Theorems A and B, we must show that when \( N_1(x) \) has
constant dimension 1, we can reduce to codimension 1.

Let \( U \) be any coordinate neighborhood of \( M^2 \). We choose orthonormal normal
vector fields \( \xi_1, \xi_2 \) on \( U \) so that \( \xi_1, \xi_2, J\xi_1, J\xi_2 \) span \( T_u(M^2) \) for any \( u \in U \) and
such that \( \xi_1, J\xi_1 \) span \( N_1(u) \) for any \( u \in U \). We then prove

**Proposition 3.** For any \( x \in U \) and \( X \in T_x(M^2) \) the following equations are true:

(i) \( \nabla_X^\perp \xi_1 = t_{11}(X)J\xi_1 \),

(ii) \( \nabla_X^\perp \xi_2 \) and \( \nabla_X^\perp J\xi_2 \) span \( \{ \xi_2, J\xi_2 \} \).

**Proof.** Equation (1.5) says that

\[
(\nabla_X A_2)Y - \sum_{\beta=1}^2 s_{\beta 2}(X)A_\beta Y - \sum_{\beta=1}^2 t_{\beta 2}(X)JA_\beta Y
\]

is symmetric in \( X \) and \( Y \).

Since \( A_2 = 0 \), \( \nabla_X A_2 = 0 \) and (1.5) can be written as

\[
(3.1) \quad s_{12}(X)A_1 Y + t_{12}(X)JA_1 Y = s_{12}(Y)A_1 X + t_{12}(Y)JA_1 X.
\]

We can choose \( X, Y \) linearly independent vectors so that \( A_1 X = \alpha X \) and \( A_1 Y = \alpha Y \),
since \( \text{dim } M = 2 \). Then (3.1) becomes

\[
(3.2) \quad s_{12}(X)\alpha Y + t_{12}(X)\alpha JY = s_{12}(Y)\alpha X + t_{12}(Y)\alpha JX.
\]

But \( X, Y, JX, JY \) are linearly independent, so (3.2) implies

\[
(3.3) \quad s_{12}(X) = t_{12}(X) = 0.
\]

A similar calculation shows that (3.3) holds for a vector \( X \), so that \( A_1 X = -\alpha X \),
and, hence, (3.3) holds for all \( X \in T_x(M^2) \). We recall that

\[
(1.1) \quad \nabla_X^\perp \xi_\beta = \sum_{\gamma=1}^2 s_{\gamma \beta}(X)\xi_\gamma + \sum_{\gamma=1}^2 t_{\gamma \beta}(X)J\xi_\gamma, \quad \beta = 1, 2.
\]

Then \( s_{\beta \gamma} = -s_{\gamma \beta} \) and \( t_{\beta \gamma} = t_{\gamma \beta} \) and (3.3) imply that for \( \beta = 1 \), (1.1) becomes

\[
\nabla_X^\perp \xi_1 = t_{11}(X)J\xi_1,
\]

proving (i). For the same reasons, for \( \beta = 2 \), (1.1) becomes

\[
(3.4) \quad \nabla_X^\perp \xi_2 = t_{22}(X)J\xi_2.
\]

Then \( \nabla_X^\perp J\xi_2 = J(\nabla_X^\perp \xi_2) \) and (3.4) proves (ii).

Proposition 3 implies \( N_1(x) \) and \( N_0(x) \) are invariant with respect to \( \nabla^\perp \).

Propositions 2 and 3 and the following lemmas complete the proofs of Theorems A
and B.

**Lemma 4 (Cecil [1]).** Let \( f: M^n \to \tilde{M}^{n+p}() \) be a Kaehler immersion of \( M^n \) into
\( \tilde{M}^{n+p}() \). Suppose \( N_1(x) \) has constant dimension \( k \) and is parallel with respect to the
normal connection. Then there is a totally geodesic \((n+k)\)-dimensional submanifold
\( \tilde{M}^{n+k}() \), such that \( f(M^n) \subset \tilde{M}^{n+k}() \).
Lemma 5 (Chern [2] and Smyth [8]). Let $M^n$ be an Einstein Kaehler hypersurface of $\tilde{M}^{n+1}(\tilde{c})$. If $n \geq 2$, then $M$ is totally geodesic in $\tilde{M}$ or $S = (n/2)\tilde{c}g$, the latter case arising only when $\tilde{c} > 0$. Moreover, the immersion is rigid.

Proof. See, for example, [6, p. 95].

Bibliography


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