ON THE UNIVERSALITY OF WORDS FOR
THE ALTERNATING GROUPS

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Abstract. We prove the following theorem on the finite alternating groups An: For each pair \((p, q)\) of nonzero integers there exists an integer \(N(p, q)\) such that, for each \(n \geq N\), any even permutation \(a \in A_n\) can be written in the form \(a = b^p \cdot c^q\) for some suitable elements \(b, c \in A_n\). A similar result is shown to be true for the finite symmetric groups \(S_n\) provided that \(p\) or \(q\) is odd.

1. Results. Let \(F\) be a free group and \(W = W(x_1, \ldots, x_n) \in F\) a word in free variables \(x_1, \ldots, x_n \in F\). For a group \(G\), we say that \(W\) is \(G\)-universal if for any \(g \in G\) the equation \(g = W\) can be solved in \(G\), i.e. there are \(g_1, \ldots, g_n \in G\) such that \(g = W(g_1, \ldots, g_n)\).

In \([14]\), Silberger asked whether for each pair \((p, q)\) of nonzero integers there exists an integer \(N(p, q)\) such that the word \(W = W(x, y) = x^p \cdot y^q\) is universal for each finite alternating group \(A_n\) with \(n \geq N(p, q)\). For \(p = q\) this was shown to be true in \([11]\). Ehrenfeucht et al. \([5]\) proved that if both \(p\) and \(q\) are not divisible by 3, then \(W\) is \(A_n\)-universal for indeed any \(n \in \mathbb{N}\).

In the following, if \(p, q\) are nonzero integers, we let \(m(p, q) \in \mathbb{N}\) denote the product of all primes dividing \(p \cdot q\), with the convention that \(m(p, q) = 1\) if \(p, q \in \{-1, 1\}\). Using a result of Bertram \([1]\), we show that Silberger’s question can in general be answered positively:

**Theorem 1.** Let \(p, q\) be nonzero integers and \(W(x, y) = x^p \cdot y^q\). Then \(W\) is \(A_n\)-universal for any \(n \geq 4m + 1\) where \(m = m(p, q)\). Moreover, for any \(n \geq \max\{4m + 1, 9\}\) there exists \(l = l(n, m) \leq n - 2\) with the following property: For any \(a \in A_n\) there are \(b, c \in A_n\) such that \(a = b^p \cdot c^q\), and \(b\) and \(c\) each consist only of one cycle of length \(l\) and \(n - 1\) fixed points; in particular, \(b\) and \(c\) are conjugate to each other in \(A_n\).

Note that the conjugacy class of the elements \(b, c\) in Theorem 1 is independent of the special choice of \(a \in A_n\).

We remark that Silberger \([15]\) also obtained, independently, that \(W = x^p \cdot y^q\) is \(A_n\)-universal for all sufficiently large \(n\). An essential improvement of the bound \(4m + 1\) is contained in a forthcoming paper by Brenner, Evans and Silberger \([2]\).
As is well known, already Ito [9] and Ore [12] showed that any element of $A_n$ ($n \geq 5$) can be expressed as a single commutator. This was reproved and strengthened by various authors, cf. e.g. Hsü Ch'eng-hao [8], Bertram [1]. As a first immediate consequence of Theorem 1 we obtain

**Corollary 1.** Let $0 \neq p \in \mathbb{Z}$, $m = m(p, p)$ and $n \geq \max\{4m + 1, 9\}$. Then there exists a conjugacy class $C$ in $A_n$ such that for any $a \in A_n$ there exists elements $c \in C$, $b \in A_n$ satisfying $a = [c^p, b]$.

Ehrenfeucht and Silberger [6] characterized all pairs $(p, q)$ of integers for which $W = x^p \cdot y^q$ is universal for each finite symmetric group $S_n (n \in \mathbb{N})$. Of course, this is the case whenever both $p$ and $q$ are odd, since any permutation is a product of two involutions. Moreover, also for some, but not all, pairs $(p, q)$ with $p$ odd and $q$ even, $W$ is $S_n$-universal for each $n \in \mathbb{N}$. For instance, for any $2 \leq n \in \mathbb{N}$ the word $x^n \cdot y^n!$ is not universal for $S_n$. This situation is different if we consider "$S_n$-universality for each sufficiently large $n \in \mathbb{N}$":

**Theorem 2.** Let $p, q$ be nonzero integers such that $p$ or $q$ is odd and $W = x^p \cdot y^q$. Then $W$ is $S_n$-universal for any $n \geq \max\{4m - 4, 6\}$ where $m = m(p, q)$. Moreover, in this case, for any $a \in S_n$ there exist $b, c \in S_n$ such that $a = b^p \cdot c^q$ and $b$ and $c$ each consist only of precisely one nontrivial cycle (and, possibly, fixed points).

Next we consider arbitrarily long and complex words which can be written as a product of three nontrivial words with disjoint sets of variables; here a word is called nontrivial if it does not reduce identically to 1. Using results of Hall [7] and Moran [10] we show

**Theorem 3.** Let $W_1 = W_1(x_1, \ldots, x_p)$, $W_2 = W_2(y_1, \ldots, y_q)$, $W_3 = W_3(z_1, \ldots, z_r)$ be three nontrivial words in free pairwise different variables $x_i$, $y_j$, $z_k$, and $W = W(x_i, y_j, z_k) = W_1 \cdot W_2 \cdot W_3$. Then there exists an $N \in \mathbb{N}$ such that, for each $n \geq N$, $A_n$ has the following property: For each $g \in A_n$ there are permutations $a_i, b_j, c_k \in A_n$ ($1 \leq i \leq p, 1 \leq j \leq q, 1 \leq k \leq r$) with orders all powers of 2 such that $g = W(a_i, b_j, c_k)$. In particular, $W$ is $A_n$-universal.

P. Hall calls a group $G$ $W$-elliptic of degree $d$, if any $g \in G$ is a product of at most $d$ $W$-elements $W(g_1, \ldots, g_n)$ with $g_1, \ldots, g_n \in G$. (Hence $W$-ellipticity of degree 1 coincides, in the present notation, with universality of $W$.) As an immediate consequence of Theorem 3 we obtain

**Corollary 2.** Let $W = W(x_1, \ldots, x_m)$ be any nontrivial word. Then, for all sufficiently large $n \in \mathbb{N}$, $A_n$ is $W$-elliptic of degree $k$ for any $k \geq 3$.

Here, as well as in Theorem 3, it remains an open problem whether the number 3 may be replaced by 2 or even by 1.

If $M$ is an infinite set, we denote by $S^0_M$ the group of all permutations of $M$ with finite support and by $A_M$ the infinite alternating group on $M$, i.e. the (simple) group of all elements of $S^0_M$ which are, if restricted to their support, even permutations. As
an obvious consequence of Theorems 1–3 we have

**Corollary 3.** Let $M$ be an infinite set.

(a) If $W$ is a word of the kind described in Theorem 1 or in Theorem 3, then $W$ is $A_M$-universal.

(b) If $W$ is a word of the kind described in Theorem 2, then $W$ is $S_M$-universal.

For related results on this topic, in particular concerning universal words for infinite symmetric groups, we refer the reader to [3, 4, 14] and the literature cited there.

**2. Proof of our results.** As usual, we let $[x]$ denote the integer part of $x \in \mathbb{Q}$.

**Proof of Theorem 1.** Since $n \geq 4m + 1$, an easy calculation shows that the interval $[[\frac{3}{2}n] - 1, n - 1]$ contains at least $m + 2$ elements, in particular some multiple $M$ of $m$. Again by $n \geq 4m + 1$ (and $n \geq 9$ if $m = 1$), we can choose an odd number $l \in \{M - 2, M - 1, M + 1, M + 2\}$ (and possibly $l = M$ if $m = 1$) with $[\frac{3}{2}n] \leq l \leq n - 2$. Note that $l$ is relatively prime with $m$. Now if $a \in A_n$, by Bertram [1] there exist two permutations $d, e \in S_n$, each consisting of precisely one cycle of length $l$ and $n - l$ fixed points, such that $a = d \cdot e$. Since $l$ is odd, we have $d, e \in A_n$, and since $l, p$ and $l, q$ are relatively prime, respectively, $d^p$ and $e^q$ each have again precisely one cycle of length $l$ and are hence conjugate to $d$ and $e$. Thus $d = b^p, e = c^q$ for some elements $b, c \in A_n$ which each have precisely one cycle of length $l$ and $n - l \geq 2$ fixed points. As is well known (cf. [13, 11.1.5]), $b$ and $c$ are conjugate to each other in $A_n$. Since $a = d \cdot e = b^p \cdot c^q$, the result follows.

The following proof uses similar ideas as the previous one:

**Proof of Theorem 2.** Assume $n \geq \max\{4m - 4, 6\}$. If $n \geq 9$, choose some multiple $M$ of $m$ in the interval $[[\frac{3}{2}n], n]$; if $n = 6 (7, 8)$, let $M = 4 (4, 6)$, respectively. Now let $a \in S_n$. If $a \in A_n$, choose $l \in \{M - 1, M + 1\}$ with $[\frac{3}{2}n] \leq l \leq n$. As in the proof of Theorem 1, we obtain $a = b^p \cdot c^q$ for some permutations $b, c \in S_n$ which have only one cycle of length $l$. Now assume $a \notin A_n$. Choose $l \in \{M - 2, M + 1\}$ such that $[\frac{3}{2}n] \leq l \leq n - 1$. We distinguish between two cases:

**Case I.** Assume $p$ is even (hence $q$ odd) and $l = M + 1$, or $p$ is odd and $l = M - 2$. In this case, by Bertram [1, Corollary 3.1] there are $d, e \in S_n$ such that $a = d \cdot e$ and $d (e)$ consists only of one cycle of length $l (l + 1)$; note that $l, p (l + 1, q)$ are relatively prime, respectively.

**Case II.** Assume $p$ is odd and $l = M + 1$, or $p$ is even (hence $q$ odd) and $l = M - 2$. Again by Bertram [1, Corollary 3.1], there are $d, e \in S_n$ such that $a = d \cdot e$ and $d (e)$ consists only of one cycle of length $l + 1 (l)$; here $l + 1, p (l, q)$ are relatively prime, respectively.

In any case we can find permutations $b, c \in S_n$, each consisting only of one nontrivial cycle of length $l$ or $l + 1$, such that $d = b^p, e = c^q$. Thus $a = d \cdot e = b^p \cdot c^q$ as claimed.

For the convenience of the reader, let us note two results of the literature which we will need for the proof of Theorem 3.
Lemma 2.1 (Hall [7, Lemma 7]). Let $W = W(x_1, \ldots, x_n)$ be any nontrivial word. Then there exists a finite 2-group $G$ with elements $a_1, \ldots, a_n \in G$ such that $a = W(a_1, \ldots, a_n) \in G$ has order 2.

Lemma 2.2 (Moran [10, Theorem 0]). Let $5 \leq n \in \mathbb{N}$ and $s \in S_n$ such that $s^2 = 1$ and $s$ has $l$ fixed points. Then each permutation in $A_n$ is a product of three conjugates of $s$ if and only if the following three conditions hold:

(1) $l \equiv n \pmod{2}$;
(2) $\frac{1}{2} \cdot (n - l) \equiv 0 \pmod{2}$;
(3) $0 < l \leq \frac{1}{2}(n + 4)$ if $n$ is even, and $0 < l \leq \frac{1}{2}(n + 2)$ if $n$ is odd.

Now we give the

Proof of Theorem 3. By Lemma 2.1, we can find a finite 2-group $G$ with $|G| \geq 4$ and elements $a_i, b_j, c_k \in G$ ($1 \leq i \leq p, 1 \leq j \leq q, 1 \leq k \leq r$) such that $a = W_1(a_i)$, $b = W_2(b_j)$, $c = W_3(c_k)$ all have order 2 in $G$. We claim that $N = 3 \cdot |G| - 5$ satisfies the assertion of the theorem. Indeed, let $n \geq N$ and $g \in A_n$. Find $m, l \in \mathbb{N}$ with $m \cdot |G| + l = n$ and $1 \leq l \leq |G|$; then $l, n$ satisfy conditions (1)–(3) of Lemma 2.2 since 4 divides $|G|$ and by our choice of $N$. Hence by Lemma 2.2 there are (conjugate) involutions $x, y, z \in S_n$, each having precisely $l$ fixed points and hence $m \cdot |G|/2$ cycles of length 2, such that $g = x \cdot y \cdot z$.

Now embed $\varphi: G \to S_n$ via Cayley's right-regular representation. Then $a_1^\varphi, b_2^\varphi, c_3^\varphi \in S_\varphi$ all have orders powers of 2 and satisfy $a_1^\varphi = W_1(a_1^\varphi), b_2^\varphi = W_2(b_2^\varphi), c_3^\varphi = W_3(c_3^\varphi)$, where $a_1^\varphi, b_2^\varphi, c_3^\varphi$ are permutations of $G$ each consisting of $|G|/2$ cycles of length 2. By taking $m$ copies of $(G, S_\varphi)$ and an additional set with $l$ elements, we can construct permutations $d, e, f, d_1, e_1, f_k \in S_n$ such that $d = W_1(d_1), e = W_2(e_1), f = W_3(f_k)$ each consist of $m \cdot |G|/2$ cycles of length 2 and $l$ fixed points and $d_1, e_1, f_k$ the same orders as $a_1^\varphi, b_2^\varphi, c_3^\varphi$, respectively, and also $l$ fixed points (in fact, we may perform our construction such that all these elements have identical fixed point sets), for $1 \leq i \leq p, 1 \leq j \leq q, 1 \leq k \leq r$. Now $x, y, z$ are conjugate to $d, e, f$, respectively. So we obtain $x = W_1(x_1), y = W_2(y_1), z = W_3(z_k)$ for suitable conjugates $x_1, y_1, z_k \in S_n$ of $d_1, e_1, f_k$, respectively; thus $x_1, y_1, z_k$ all have orders powers of 2. Hence $g = x \cdot y \cdot z = W(x_1, y_1, z_k)$, as claimed.

References
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