ON WEIGHTED INTEGRABILITY OF TRIGONOMETRIC SERIES AND L^1-CONVERGENCE OF FOURIER SERIES

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ABSTRACT. A result concerning integrability of \( f(x)L(1/x)g(x)L(1/x) \), where \( f(x)(g(x)) \) is the pointwise limit of certain cosine (sine) series and \( L(x) \) is slowly vary in the sense of Karamata [5] is proved. Our result is an excluded case in more classical results (see [4]) and also generalizes a result of G. A. Fomin [1]. Also a result of Fomin and Telyakovskii [6] concerning \( L^1 \)-convergence of Fourier series is generalized. Both theorems make use of a generalized notion of quasi-monotone sequences.

1. Introduction. A classical problem in the theory of trigonometric series concerns sufficient conditions in terms of the coefficients \( \{a(n)\} \) for the Fourier character of cosine series

\[
\frac{a(0)}{2} + \sum_{n=1}^{\infty} a(n) \cos nx
\]

and the conjugate or sine series

\[
\sum_{n=1}^{\infty} a(n) \sin nx.
\]

All known results employ conditions which imply that the null sequence \( \{a(n)\} \) is of bounded variation (\( \sum_{n=1}^{\infty} |\Delta a(n)| < \infty \), \( \Delta a(n) = a(n) - a(n + 1) \), and \( a(n) = o(1) \) (\( n \to \infty \)). This further implies that the pointwise limit of (1.1) and (1.2) exist on \( (0, \pi] \); these are denoted \( f(x) \) and \( g(x) \), respectively. Consequently, for the Fourier character of (1.1) or (1.2) it is necessary and sufficient that \( f \) or \( g \) be Lebesgue integrable on \( (0, \pi] \). A recent result in this direction is the following, due to Fomin (see also [2, 3]).

**Theorem 1.1.** Let \( a(n) = o(1) \) (\( n \to \infty \)), and for some \( p > 1 \) let

\[
\sum_{n=1}^{\infty} \left( \frac{\sum_{k=n}^{\infty} |\Delta a(k)|^p}{n} \right)^{1/p} < \infty.
\]

Then

(i) \( f \in L^1(0, \pi] \), and

(ii) \( g \in L^1(0, \pi] \) if and only if \( \sum_{n=1}^{\infty} |a(n)|/n < \infty \).


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Many authors have varied the point of view of the above problem by considering weighted integrability of the sum functions (see the monograph of Boas [4] for a survey). These results give criteria for the integrability of $x^{-\gamma}f(x) L(1/x)$ and $x^{-\gamma}g(x) L(1/x)$, where $\gamma > 0$ and $L(\cdot)$ is a slowly varying function in the sense of Karamata [5]. In §3 an integrability result for $f(x) L(1/x)$ and $g(x) L(1/x)$ is proved, generalizing Theorem 1.1 in the case of cosine series and giving a restricted generalization in the case of sine series.

In the final section a generalization of the following theorem due to Fomin and Telyakovskii [6] (see also [7]) is proved. For succinct formulation the $n$th partial sums of the Fourier series of $f \in L^1(0, \pi)$ are denoted $S_n(f) = S_n(f, x)$. Also recall (Szasz [8]) that a null sequence $\{a(n)\}$ is said to be quasi-monotone if, for some $\alpha > 0$, $a(n)/n^\alpha \downarrow$ for $n \geq n_0(\alpha)$.

**Theorem 1.2.** Let (1.1) be the Fourier series of $f \in L^1(0, \pi)$ with quasi-monotone coefficients. Then $\|S_n(f) - f\| = o(1) \ (n \to \infty)$ if and only if $a(n) \log n = o(1) \ (n \to \infty)$.

An analogous result holds for sine series. Both our results make use of a generalization of quasi-monotone sequence developed in §2.

2. Preliminaries. A positive measurable function $L(u)$ is said to be slowly varying in the sense of Karamata [5] if, for $\lambda > 0$,

\[
\lim_{u \to +\infty} \frac{L(\lambda u)}{L(u)} = 1.
\]

Karamata [5] proved that (2.1) holds uniformly for $\lambda$ contained in a bounded closed interval. Slowly varying sequences are defined analogously: a positive sequence $\{l(n)\}$ is said to be slowly varying if, for $\lambda > 0$,

\[
\lim_{n \to +\infty} \frac{l(\lambda n)}{l(n)} = 1.
\]

The class of slowly varying functions (sequences) is denoted by $SV(\mathbb{R})$ ($SV(\mathbb{N})$).

In [9] Karamata introduced regularly varying sequences: a positive sequence $\{r(n)\}$ is said to be regularly varying if, for $\lambda > 0$ and some $\alpha > 0$,

\[
\lim_{n \to \infty} \frac{r(\lambda n)}{r(n)} = \lambda^\alpha.
\]

The class of such is denoted by $RV(\mathbb{N})$. Regularly varying sequences are characterized [9] in form as follows: $\{r(n)\) \in RV(\mathbb{N})$ if and only if $r(n) = n^\alpha l(n)$, for some $\alpha > 0$ and some $\{l(n)\) \in SV(\mathbb{N})$.

A null sequence $\{a(n)\)$ is said to be regularly varying quasi-monotone if for some $\{r(n)\) \in RV(\mathbb{N})$, $a(n)/r(n) \downarrow$ for $n \geq n_0$. The class of such sequences is denoted $RQM$ and properly contains quasi-monotone sequences. We can now give the following generalization of the Cauchy condensation test.

**Lemma 2.1.** Let $\{a(n)\) \in RQM$. Then the series $\sum_{n=1}^{\infty} a(n) l(n)$ and the series $\sum_{n=0}^{\infty} 2^n a(2^n) l(2^n)$ are equiconvergent for every $\{l(n)\) \in SV(\mathbb{N})$. 

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Proof. For sufficiently large $k$, $a(k)/k^\alpha l_1(k) \downarrow$, where $\alpha > 0$ and $\{l_1(k)\} \in \text{SV}(N)$. Consequently, for sufficiently large $n$,

\[
\frac{a(2^n + 1)}{2^{\alpha(n+1)} l_1(2^{n+1})} \sum_{k=2^n}^{2^{n+1}-1} k^\alpha l_1(k) l(k) \leq \sum_{k=2^n}^{2^{n+1}-1} a(k) l(k) \\
\leq \frac{a(2^n)}{2^{\alpha n} l_1(2^n)} \sum_{k=2^n}^{2^{n+1}-1} k^\alpha l_1(k) l(k),
\]

and so,

\[
\frac{1}{2^\alpha} \frac{a(2^n + 1)}{l_1(2^{n+1})} \sum_{k=2^n}^{2^{n+1}-1} l_1(k) l(k) \leq \sum_{k=2^n}^{2^{n+1}-1} a(k) l(k) \\
\leq 2^\alpha \frac{a(2^n)}{l_1(2^n)} \sum_{k=2^n}^{2^{n+1}-1} l_1(k) l(k).
\]

Since $\{l_1(k)l(k)\} \in \text{SV}(N)$, the aforementioned uniform nature of (2.1) or (2.2) gives

\[
\sum_{k=2^n}^{2^{n+1}-1} l_1(k) l(k) \sim 2^n l_1(2^n) l(2^n) \quad (n \to \infty),
\]

from which the conclusion follows.

Another basic property of slowly varying functions is the asymptotic relation [5]

\[
u^\alpha \max_{u \leq s < \infty} s^{-\alpha} L(s) \sim L(u) \quad (u \to \infty),
\]

for any $\alpha > 0$. The following lemma resembles a classical Abelian theorem [10].

**Lemma 2.2.** Let $\{l(n)\} \in \text{SV}(N)$ and let $\{m_k\}_0^\infty$ be a positive sequence such that, for some $0 < \alpha < 1$,

\[
\sum_{k=n}^{\infty} \frac{1}{m_k^{1-\alpha}} = O \left( \frac{1}{m_n^{1-\alpha}} \right) \quad (n \to \infty).
\]

Then

\[
\sum_{k=0}^{\infty} \frac{l(m_k)}{m_k} < \infty
\]

and

\[
\sum_{k=n}^{\infty} \frac{l(m_k)}{m_k} = O \left( \frac{l(m_n)}{m_n} \right) \quad (n \to \infty).
\]

Proof. For $N > n$, 

\[
\sum_{k=n}^{N} \frac{l(m_k)}{m_k} \leq \left( \sup_{k \geq n} m_k^{-\alpha} l(m_k) \right) \sum_{k=n}^{\infty} \frac{1}{m_k^{1-\alpha}}.
\]
Consequently (2.3) holds, and
\[ \sum_{k=n}^{\infty} \frac{l(m_k)}{m_k} \leq A \left( m_n^a \sup_{k \geq n} \frac{a(l(m_k))}{m_k} \right) \frac{1}{m_n}, \]
where \( A \) is an absolute constant. This completes the proof.

3. Weighted integrability theorem. We prove the following theorem.

**Theorem 3.1.** Let \( L \in SV(\mathbb{R}) \) such that \( L(u) \to \infty \) \((u \to \infty)\), let \( a(n) = o(1) \) \((n \to \infty)\), and for some \( p > 1 \), let
\[ \sum_{n=1}^{\infty} L(n) \left( \frac{\sum_{k=n}^{\infty} |\Delta a(k)|^p}{n} \right)^{1/p} < \infty. \]
Then (i) \( f(x)L(1/x) \in L^1(0, \pi) \), and
(ii) if \( \{ |a(n)| \} \in RQM \), then \( g(x)L(1/x) \in L^1(0, \pi) \) if and only if
\[ \sum_{n=1}^{\infty} \frac{|a(n)|}{n} L(n) < \infty. \]

**Proof.** Applying Lemma 2.1 and the methods of [2], the series in (3.1) is equiconvergent with
\[ \sum_{n=0}^{\infty} 2^n L(2^n) \left( \frac{1}{2^n} \sum_{k=2^n}^{2^{n+1}-1} |\Delta a(k)|^p \right)^{1/p}. \]
By Jensen's inequality, (3.1) implies \( \sum_{n=1}^{\infty} |\Delta a(n)| L(n) < \infty \), so that \( \{ a(n) \} \) is of bounded variation. Also, we may suppose \( 1 < p \leq 2 \), a necessary technicality. We prove (ii); (i) is similar. Summation by parts yields the pointwise limit
\[ g(x) = \sum_{n=1}^{\infty} \Delta a(n) \tilde{D}_n(x), \quad x \in (0, \pi], \]
where
\[ \tilde{D}_n(x) = \frac{\cos(x/2) - \cos(n + 1/2)x}{2 \sin(x/2)} \]
is the conjugate Dirichlet kernel. Letting \( a(0) = 0 \) and
\[ \tilde{D}_n(x) = -\frac{\cos(n + 1/2)x}{2 \sin(x/2)}, \]
we may write
\[ g(x) = \sum_{n=0}^{\infty} \Delta a(n) \tilde{D}_n(x), \quad x \in (0, \pi]. \]
The result will be obtained by means of the following estimate: for \( N = 1, 2, \ldots \),
\[ \int_{\pi/2}^{\pi} |g(x)|L\left(\frac{1}{x}\right) dx = \sum_{n=0}^{N} |a(2^n)| \int_{\pi/2}^{\pi/2+n} L\left(\frac{1}{x}\right) \frac{dx}{x} \]
\[ + O \left( \sum_{n=0}^{\infty} 2^n L(2^n) \left( \frac{1}{2^n} \sum_{k=0}^{\infty} |\Delta a(k)|^p \right)^{1/p} \right). \]
the O-term being uniform with respect to $N$. It follows that $g(x)L(1/x) \in L^1(0, \pi)$ if and only if

$$\sum_{n=0}^{\infty} |a(2^n)| \frac{1}{\pi} \int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} L\left(\frac{1}{x}\right) \frac{dx}{x} < \infty. $$

A change of variables gives

$$\int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} L\left(\frac{1}{x}\right) \frac{dx}{x} = \int_{\pi}^{2^n} L(2^n u) \frac{du}{u} \sim (\log 2) L(2^n) \quad (n \to \infty).$$

Thus, the series in (3.5) is equiconvergent with $\sum_{n=0}^{\infty} |a(2^n)| L(2^n)$, completing the proof by Lemma 2.1, provided we verify (3.4). For $N = 1, 2, \ldots$,

$$\int_{\pi 2^{-N+1}}^{\pi 2^{-N}} |g(x)|L\left(\frac{1}{x}\right) dx - \sum_{n=0}^{N} \int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} \left| \sum_{k=0}^{2^{n}-1} \Delta a(k) \overline{D}_k(x) \right| L\left(\frac{1}{x}\right) dx$$

Denote the right side by $I_N$; applying Hölder's inequality ($1/p + 1/q = 1$), followed by the Riesz [11] extension of the Hausdorff-Young theorem, one obtains

$$I_N \leq \frac{1}{2} \sum_{n=0}^{N} \left( \int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} L^p\left(\frac{1}{x}\right) \frac{dx}{x^p} \right)^{1/p} \left( \sum_{k=2^n}^{\infty} \sum_{k=2^n}^{\infty} \Delta a(k) \cos \left( k + \frac{1}{2} \right) \right)^{1/p},$$

where $\| \cdot \|_q$ is the $L^q(0, \pi)$-norm, and $A_p$ is a constant dependent only on $p$. As in (3.6),

$$\int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} L^p\left(\frac{1}{x}\right) \frac{dx}{x^p} \leq B 2^{n(p-1)} L^p(2^n),$$

where $B$ is an absolute constant. From (3.8),

$$I_N \leq A_p \sum_{n=0}^{N} 2^n L(2^n) \left( \frac{1}{2^n} \sum_{k=2^n}^{\infty} |\Delta a(k)|^p \right)^{1/p},$$

where $B$ has been absorbed into $A_p$. Returning to (3.7), we have

$$\int_{\pi 2^{-N+1}}^{\pi 2^{-N}} |g(x)|L^p\left(\frac{1}{x}\right) dx = \sum_{n=0}^{N} \int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} \left| \sum_{k=0}^{2^{n}-1} \Delta a(k) \overline{D}_k(x) \right| L\left(\frac{1}{x}\right) dx + O\left( \sum_{n=0}^{\infty} 2^n L(2^n) \left( \frac{1}{2^n} \sum_{k=2^n}^{\infty} |\Delta a(k)|^p \right)^{1/p} \right),$$

uniformly in $N$. Denote the first term on the right side by $J_N$. Applying the uniform estimate

$$|\overline{D}_n(x) + 1/x| \leq A(n + 1), \quad x \in (0, \pi].$$
\( A \) being an absolute constant, we have

\[
J_N - \sum_{n=0}^{N} \int_{\pi 2^{-n+1}}^{\pi 2^{-n}} \left| \sum_{k=0}^{2^n-1} \Delta a(k) L \left( \frac{1}{x} \right) \frac{dx}{x} \right|
\]

\[
\leq A \sum_{n=0}^{N} \int_{\pi 2^{-n+1}}^{\pi 2^{-n}} \sum_{k=0}^{2^n-1} |\Delta a(k)|(k+1) L \left( \frac{1}{x} \right) dx.
\]

Again similar to (3.6),

\[
\int_{\pi 2^{-n+1}}^{\pi 2^{-n}} L \left( \frac{1}{x} \right) dx \sim \frac{L(2^n)}{2^{n+1}} = O \left( \frac{L(2^n)}{2^n} \right) \quad (n \to \infty).
\]

Consequently, denoting the right side of (3.10) by \( J_N' \) and absorbing all absolute constants into \( A \), we get

\[
J_N' \leq A \sum_{n=0}^{N} \frac{L(2^n)}{2^n} \sum_{k=0}^{2^n-1} |\Delta a(k)|(k+1)
\]

\[
\leq A \sum_{k=0}^{2^N-1} |\Delta a(k)|(k+1) \sum_{n=[\log_2(k+1)]}^{N} \frac{L(2^n)}{2^n},
\]

\([\log_2(k+1)]\) denoting the greatest integer in the base two logarithm of \( k+1 \). Appealing to Lemma 2.2 one obtains

\[
J_N' \leq A \sum_{k=0}^{2^N-1} |\Delta a(k)|(k+1) \sum_{n=[\log_2(k+1)]}^{\infty} \frac{L(2^n)}{2^n}
\]

\[
\leq A \sum_{k=0}^{2^N-1} |\Delta a(k)|(k+1) \frac{L(k+1)}{k+1}
\]

\[
\leq A \sum_{n=0}^{\infty} 2^n L(2^n) \left( \frac{1}{2^n} \sum_{k=2^n}^{\infty} |\Delta a(k)|^p \right)^{1/p}.
\]

Returning to (3.10), we get

\[
J_N = \sum_{n=0}^{N} |a(2^n)| \int_{\pi 2^{-n+1}}^{\pi 2^{-n}} L \left( \frac{1}{x} \right) \frac{dx}{x} + O \left( \sum_{n=0}^{\infty} 2^n L(2^n) \left( \frac{1}{2^n} \sum_{k=2^n}^{\infty} |\Delta a(k)|^p \right)^{1/p} \right),
\]

which concludes the proof of (3.4).

4. \( L^1 \)-convergence of Fourier series. In this section we prove the following theorem concerning \( L^1 \)-convergence of Fourier cosine series, an analogous result holds for Fourier sine series.

**Theorem 4.1.** Let (1.1) be the Fourier series of some \( f \in L^1(0, \pi) \) with \( \{a(n)\} \in \text{RQM} \). Then \( ||S_n(f) - f|| = o(1) \quad (n \to \infty) \) if and only if \( a(n) \log n = o(1) \quad (n \to \infty) \).
Proof. Let $\sigma_n(f) = \sigma_n(f, x)$ denote the $(C, 1)$ means of the Fourier cosine series of $f$. Summation by parts yields

$$S_n(f, x) - \sigma_n(f, x) = \frac{1}{n+1} \sum_{k=1}^{n} k a(k) \cos kx$$

$$= \frac{1}{n+1} \sum_{k=1}^{n-1} \Delta(a(k)) \left[ D_k(x) - \frac{1}{2} \right]$$

$$+ \frac{n}{n+1} a(n) \left[ D_n(x) - \frac{1}{2} \right],$$

where

$$D_n(x) = \frac{1}{2} + \sum_{k=0}^{n} \cos kx = \frac{\sin(n + 1/2)x}{2 \sin(x/2)}$$

is the Dirichlet kernel. Rearranging terms gives the following useful identity:

$$S_n(f, x) - \sigma_n(f, x) = \frac{1}{n+1} \sum_{k=1}^{n-1} k \Delta a(k) D_k(x)$$

$$- \frac{n}{n+1} \sum_{k=0}^{n-1} a(k + 1) D_k(x) + a(n) D_n(x).$$

Applying the $L^1$-norm and using the well-known estimate

$$\|D_n\| = (2/\pi) \log n + O(1) \quad (n \to \infty),$$

we get

(4.1) $\|S_n(f) - \sigma_n(f)\| \leq \frac{2}{\pi} \cdot \frac{1}{n+1} \sum_{k=0}^{n-1} k |\Delta a(k)| \log k + \frac{2}{\pi} \cdot \frac{1}{n+1} \sum_{k=0}^{n-1} |a(k + 1)| \log k$

$$+ \frac{2}{\pi} a(n) \log n + o(1) \quad (n \to \infty).$$

For sufficiency, the hypothesis $a(n) \log n = o(1) \quad (n \to \infty)$ implies that the second and third terms on the right side are $o(1) \quad (n \to \infty)$. Hence, we must show that

(4.2) $\frac{1}{n+1} \sum_{k=1}^{n} k |\Delta a(k)| \log n = o(1) \quad (n \to \infty).$

Since $\{a(n)\} \in \text{RQM}$, for some $\alpha > 0$ and some $\{l(n)\} \in \text{SV}(N)$, we have $a(n)/n^{\alpha} l(n) \downarrow$. This implies that

$$a(n + 1) \leq (1 + \alpha/n) \frac{l(n + 1)}{l(n)} a(n),$$

without loss of generality, for all $n$. Consequently,

$$\Delta a(n) + \left[ (1 + \frac{\alpha}{n}) \frac{l(n + 1)}{l(n)} - 1 \right] a(n) \geq 0$$

and, finally,

(4.3) $|\Delta a(n)| \leq \Delta a(n) + 2 \left[ (1 + \frac{\alpha}{n}) \frac{l(n + 1)}{l(n)} \right] a(n).$
We apply (4.3) to estimate the expression in (4.2); i.e.,

\[
\frac{1}{n+1} \sum_{k=1}^{n} k|\Delta a(k)|\lg n \leq \frac{1}{n+1} \sum_{k=1}^{n} k\Delta a(k)\lg k + \frac{2}{n+1} \sum_{k=1}^{n} \left(1 + \frac{\alpha}{k}\right) \frac{l(k+1)}{l(k)} a(k)\lg k.
\]

The second term is \(o(1)\) \((n \to \infty)\) since \(\{l(n)\} \in SV(N)\) and \(a(n)\lg n = o(1)\) \((n \to \infty)\). For the first we apply summation by parts:

\[
\frac{1}{n+1} \sum_{k=1}^{n} k\Delta a(k)\lg k = \frac{1}{n+1} \sum_{k=1}^{n-1} k\log\left(1 + \frac{1}{k}\right) a(k + 1)
+ \frac{1}{n+1} \sum_{k=1}^{n-1} a(k + 1)\lg(k + 1) - \frac{n}{n+1} a(n + 1)\lg n.
\]

The first term is \(o(1)\) \((n \to \infty)\) since \(\lg(1 + 1/n) \approx 1/n\) \((n \to \infty)\); the second and third terms are \(o(1)\) \((n \to \infty)\) since \(a(n)\lg n = o(1)\) \((n \to \infty)\). This concludes the proof of sufficiency. For necessity we use the known estimate [6]

\[
\| S_n(f) - f \| \geq \sum_{k=1}^{n} \frac{a(n + k)}{k}.
\]

From the fact that \(\{a(n)\} \in RQM\) we obtain the inequality

\[
\sum_{k=1}^{n} \frac{a(n + k)}{k} = \sum_{k=1}^{n} \frac{a(n + k)}{(n + k)^\alpha l(n + k)} \cdot \frac{(n + k)^\alpha l(n + k)}{k} \\
\geq \frac{a(2n)}{(2n)^\alpha l(2n)} \sum_{k=1}^{n} \frac{(n + k)^\alpha l(n + k)}{k} \\
\geq \left(\frac{n + 1}{2n}\right)^\alpha a(2n) \frac{l(2n)}{l(2n)} \sum_{k=1}^{n} \frac{l(n + k)}{k}.
\]

The asymptotic relation \(l(k) \sim k^\beta\) \([\sup_{m \geq k} m^{-\beta} l(m)]\) \((k \to \infty)\), gives for large \(n\),

\[
\sum_{k=n+1}^{2n} \frac{l(k)}{k - n} \approx \sum_{k=n+1}^{2n} \frac{k^\beta \sup_{m \geq k} m^{-\beta} l(m)}{k - n} \\
\approx \left[ \sup_{m \geq 2n} m^{-\beta} l(m) \right] \sum_{k=n+1}^{2n} \frac{k^\beta}{k - n} \geq (n + 1)^\beta \left[ \sup_{m \geq 2n} m^{-\beta} l(m) \right] \sum_{k=1}^{n} \frac{1}{k} \\
= \left(\frac{n + 1}{2n}\right)^\beta (2n)^\beta \left[ \sup_{m \geq 2n} m^{-\beta} l(m) \right] \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2^\beta} l(2n)\lg n.
\]

Returning to (4.4) concludes the proof.

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