A NOTE ON IDEALS OF OPERATORS

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Abstract. An ideal $\mathfrak{I}$ of $\mathcal{L}(\mathcal{H})$ is said to be multiplicatively prime if $AXB \in \mathfrak{I}$ for all $X \in \mathcal{L}(\mathcal{H})$ implies $A$ or $B$ is in $\mathfrak{I}$. The only normable multiplicatively prime ideals are $\{0\}$ and $\mathcal{K}$, the compacts. Multiplicative primeness is related to other properties an ideal may possess.

Considering the following properties that an ideal $\mathfrak{I}$ of operators on separable Hilbert space may possess. (Here, ideal means two-sided and selfadjoint, but not closed.)

Definition 1. An ideal $\mathfrak{I}$ has the square root property if, for all $A \in \mathfrak{I}$, we have $\sqrt{|A|} \in \mathfrak{I}$. (As usual $|A| = \sqrt{A^*A}$.)

We note that $A \in \mathfrak{I}$ if and only if $|A| \in \mathfrak{I}$, by polar decomposition [4, p. 69].

Definition 2. An ideal $\mathfrak{I}$ is square if $\mathfrak{I} = \mathfrak{I}^2$, i.e., every element $A \in \mathfrak{I}$ can be written $A = BC$ where $B$ and $C$ are in $\mathfrak{I}$.

Definition 3. An ideal $\mathfrak{I}$ is multiplicatively prime if $AXB \in \mathfrak{I}$ for all $X \in \mathcal{L}(\mathcal{H})$ implies $A$ or $B$ is in $\mathfrak{I}$.

We remark that if dim $\mathcal{H} \geq 2$, no proper two-sided ideal in $\mathcal{L}(\mathcal{H})$ is prime in the classical algebraic sense, i.e., $AB \in \mathfrak{I}$ implies $A$ or $B$ in $\mathfrak{I}$; just let

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & X \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} Y & 0 \\ 0 & B_1 \end{bmatrix}$$

where $A_1$, $B_1$ are in $\mathfrak{I}$, but $X$ and $Y$ are not. For finite dimensions, $\mathfrak{I} = \{0\}$; for infinite dimensions, it is an easy argument using s-numbers and ideal sets, cf. below, to see that $X$ not in $\mathfrak{I}$ implies $A_1 \oplus X$ not in $\mathfrak{I}$.

It is evident that the improper ideal $\mathcal{L}(\mathcal{H})$ satisfies all three properties. We intend to find all the others. As a preliminary, we note that every proper two-sided ideal in $\mathcal{L}(\mathcal{H})$ is a subset of $\mathcal{K}$, the ideal of compact operators. For a compact operator $T$, the sequence of s-numbers of $T$, $s(T)$, is defined as follows: Let $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$ be the eigenvalues of $|T|$; then $s(T) = \{s_n(T) = \alpha_n\} = s(|T|)$.

An ideal set $I$ is a collection of sequences of real numbers $\{a_n\}_{n=1}^\infty$ with the following properties:

(i) if $\{a_n\} \in I$, then $a_n \geq 0$ for all $n$, and $\lim_{n} a_n = 0$;
(ii) if $\{a_n\} \in I$ and $\pi$ is any permutation of the positive integers then $\{a_{\pi(n)}\} \in I$;
(iii) if $\{a_n\} \in I$ and $\{b_n\} \in I$, then $\{a_n + b_n\} \in I$;
(iv) if $\{a_n\} \in I$ and $0 \leq b_n \leq a_n$ for all $n$, then $\{b_n\} \in I$.

Received by the editors September 10, 1984 and, in revised form, February 4, 1985.
1980 Mathematics Subject Classification. Primary 47D25.

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0002-9939/86 $1.00 + .25 per page

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It is a classical result of Calkin [2] that there is a bijection between ideal sets and proper two-sided ideals of operators. We also have the following [3, Lemma 1.1], see also [2]:

**LEMA.** Let $\mathcal{J}$ be a proper two-sided ideal of $\mathcal{L}(\mathcal{H})$. A compact operator $T$ belongs to $\mathcal{J}$ if and only if $s(T)$ belongs to the ideal set of $\mathcal{J}$.

An ideal $\mathcal{J}$ is said to be a **norm ideal** if there is a norm $\| \cdot \|_\mathcal{J}$ on $\mathcal{J}$ with the following properties:

(i) $(\mathcal{J}, \| \cdot \|_\mathcal{J})$ is a Banach space;
(ii) $\|STR\|_\mathcal{J} \leq \|S\| \|T\|_\mathcal{J} \|R\|$ for all $R, S \in \mathcal{L}(\mathcal{H})$, for all $T \in \mathcal{J}$;
(iii) $\|T\|_\mathcal{J} = \|T\|$ for $T$ of rank one.

The canonical examples of norm ideals are the Schatten ideals $C_p$, $1 \leq p \leq \infty$, where $\|T\|_p$ is the $l^p$ norm of $s(T)$.

**THEOREM 1.** Consider the following properties of an ideal $\mathcal{J}$:

1. $\mathcal{J}$ is multiplicatively prime;
2. $\mathcal{J}$ has the square root property;
3. $\mathcal{J}$ is square.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

**PROOF.** (1) $\Rightarrow$ (2). Suppose $A \in \mathcal{J}$; hence $|A| \in \mathcal{J}$. Look at the mapping $X \rightarrow \sqrt{|A|} X \sqrt{|A|}$ for $X \in \mathcal{L}(\mathcal{H})$. We claim the range of this mapping is in $\mathcal{J}$; if so, then by the property of being multiplicatively prime, we have $\sqrt{|A|} \in \mathcal{J}$.

But $X \rightarrow CXD$ has range in $\mathcal{J}$ if and only if $s(C)s(D)$ belongs to $I$, the ideal set of $\mathcal{J}$, by [3, Lemma 5.4 and 5.5]. Then $s(\sqrt{|A|})s(\sqrt{|A|}) = s(|A|) \in I$ by our earlier lemma, completing the proof.

(2) $\Rightarrow$ (3). The polar decomposition writes $A = U|A| = (U\sqrt{|A|})\sqrt{|A|}$ as a product of elements in $\mathcal{J}$.

(3) $\Rightarrow$ (2). This follows from Theorem 2.1 of [6].

**REMARK.** Let $\mathcal{J}_p = \bigcup_{K>0} C_{p,K}$ where $p > 1$; then $\mathcal{J}$ has the square root property, for if $\sum |\beta_j|^2 < \infty$, then $\sum (|\beta_j|)^{2r} < \infty$; thus if $A \in C_r \subseteq \mathcal{J}_p$ then $\sqrt{|A|} \in C_{2r} \subseteq \mathcal{J}_p$.

On the other hand, $\mathcal{J}$ is not multiplicatively prime, since there are compact operators $A$ and $B$ such that $A, B \notin C_p$ for every $p, 1 \leq p < \infty$, but such that $X \rightarrow AXB$ takes values in $C_1$ [3, Example 5.8].

**PROPOSITION 2.** The ideals $0, \mathcal{J}$, and $\mathcal{H}$ are multiplicatively prime.

**PROOF.** For $\mathcal{H}$, this is Proposition 4.1 of [3].

If $AXB = 0$ for all $X$ with $A, B \neq 0$, let $u$ be so that $Bu = v \neq 0$; and let $w$ be so that $Aw \neq 0$. Then for $X$ the rank-one operator $X(e) = \langle e, v \rangle w$, we have $(AXB)(u) \neq 0$.

If $AXB$ is finite rank for all $X$, but $A, B$ are not finite rank, choose $\{e_i\}_{i=1}^\infty$ such that $\{Be_i\}_{i=1}^\infty$ is an orthonormal basis for the range of $B$; and choose $\{f_i\}_{i=1}^\infty$ an orthonormal set so that $\{Af_i\}$ is a basis for the range of $A$. Then for the partial isometry $X: Be_i \rightarrow f_i$, we have $AXB$ is not finite rank.
Theorem 3. If \( \mathcal{I} \) is a proper norm ideal different from \( \mathcal{L}(\mathcal{H}) \), the following are equivalent:

1. \( \mathcal{I} \) is multiplicatively prime,
2. \( \mathcal{I} \) has the square root property,
3. \( \mathcal{I} \) is square,
4. \( \mathcal{I} = \{0\} \) or \( \mathcal{I} = \mathcal{H} \).

Proof. By Theorem 1 and Proposition 2, the only implication we need to prove is (3) \( \Rightarrow \) (4). But this is precisely the content of [7, Theorem 2.9].

Remarks. The above results lend to the following question: Which ideals \( \mathcal{I} \) of \( \mathcal{L}(\mathcal{H}) \) are multiplicatively prime?

To see if \( \mathcal{I} \) is multiplicatively prime, we ask whether \( AXB \in \mathcal{I} \) for all \( X \in \mathcal{L}(\mathcal{H}) \) implies that \( A \) or \( B \) is in \( \mathcal{I} \).

From [3, Lemma 5.1 and Corollary 5.2], we obtain the facts that if either \( A \) or \( B \) is not compact the only way for \( AXB \) to be in \( \mathcal{I} \) for all \( X \) in \( \mathcal{L}(\mathcal{H}) \) is for \( B \) (for \( A \) respectively) to be in \( \mathcal{I} \).

From Theorem 3, we see that norm ideals cannot be square, and thus cannot be multiplicatively prime. This also follows from Theorem 7.11 in [7], which states that if \( \mathcal{I} \subseteq \mathcal{H} \) is any norm ideal, there are ideals \( \mathcal{I}_1, \mathcal{I}_2 \) with \( \mathcal{I} \subseteq \mathcal{I}_k \) and \( \mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2 \). If we choose \( A_k \in (\mathcal{I}_k/\mathcal{I}) \), then we have for all \( X \) in \( \mathcal{L}(\mathcal{H}) \) that \( A_1XA_2 \in \mathcal{I}_1 \cap \mathcal{I}_2 = \mathcal{I} \). From [3, 5.4] we have \( s(A_1)s(A_2) \subseteq I \), but \( s(A_1) \not\subseteq I \) since \( A_1 \not\in \mathcal{I} \).

We conjecture that, among nonnorm ideals, only \( \mathcal{I} \) is multiplicatively prime.

References


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