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EXISTENCE OF SOLUTIONS OF $x'' + x + g(x) = p(t)$, $x(0) = 0 = x(\pi)$

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ABSTRACT. We obtain criteria for the existence of solutions of $x'' + x + g(x) = p(t)$, $x(0) = 0 = x(\pi)$, where $g\colon R \to R$ is not necessarily bounded and does not necessarily have proper limits $g(\infty)$ and $g(-\infty)$.

1. In this paper we consider the boundary value problem

$$x'' + x + g(x) = p(t), \qquad x(0) = x(\pi) = 0, \tag{1}$$

where $g\colon R \to R$ is not necessarily bounded. In the case when g is bounded this problem has been extensively studied in the literature. We study here the case when g is bounded only from one side. When g satisfies this growth hypothesis, the periodic boundary value problem has been studied in **[6]** and it is remarked **[7]** that it is not clear that the methods can be adapted for the Dirichlet problem. A particular situation of (1.1), viz., $g(x) = -\alpha x^-$, where $\alpha > 0$ and $x^-(t) = \max\{-x(t), 0\}$, has been studied in **[3]** and an open problem was raised as to whether $\int_0^\pi p(t)\sin t\,dt \leqslant 0$ is necessary and sufficient for the solvability of (1) when $g(x) = -\alpha x^-$. This was answered affirmatively in **[1]** by techniques very particular to (1) in that the nature of g is critically utilised. We consider here problem (1) from a rather general point of view wherein the above mentioned results are obtained as particular cases but one can handle more general nonlinearities as will be illustrated by examples.

2. We assume that $g\colon R \to R$ is continuous, $p \in C[0, \pi]$ and further that

(2) (i) $g(x) \geqslant 0$, $x \in R$;
(ii) $\exists$ real numbers $\rho_1 < \rho_2$ and a positive number δ such that

$$\int_0^\pi g(\rho_1 \sin t + x_1(t))\sin t\,dt \leqslant \int_0^\pi p \sin t\,dt \leqslant \int_0^\pi g(\rho_2 \sin t + x_1(t))\sin t\,dt \tag{3}$$

for all $x_1(t)$ such that $\int_0^\pi x_1(t)\sin t\,dt = 0$ and $\|x_1\|_\infty \leqslant \delta$.
(iii) $\delta > 0$ further satisfies

$$\frac{2K}{\pi}\left[\int_0^\pi |p|\sin t\,dt + \int_0^\pi p(t)\sin t\,dt\right] < \delta, \tag{4}$$

where K is defined by $(*)$ below.

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As in [**2**], the nonlinear problem (1) is equivalent to the operator equation

$$x_1 = H(I-P)Nx + PNx, \tag{5}$$

where:

(a) $N: C[0,\pi] \to Z$ is the Nemytskii operator generated by $-g(x)+p(t)$ and Z is the space of all continuous functions f: $[0,\pi] \to R$ with $\|f\|_Z = \int_0^\pi |f(t)| \sin t\, dt$;

(b) $P: L_\infty[0,\pi] \to L_\infty[0,\pi]$ is defined by

$$Px = \frac{2}{\pi}\left[\int_0^\pi x(t)\sin t\, dt\right]\sin t;$$

(c) $x(t) = x_0 \sin t + x_1(t)$, with $\int_0^\pi x_1(t)\sin t\, dt = 0$;

(d) H is the linear operator defined as follows: $\forall f_1(t)$ such that $\int_0^\pi f_1(t)\sin t\, dt = 0$, let $x_1(t)$ be the solution of

$$x_1'' + x_1 = f_1(t),$$
$$x_1(0) = x(\pi) = 0, \qquad \int_0^\pi x_1 \sin t\, dt = 0.$$

Denoting by H the transformation $f_1(t) \to Hf_1 = x_1(t)$, we can obtain the operator $H(I-P)$ (appropriately extended): $z \to C[0,\pi]$ satisfying

$$\|x_1\|_\infty \leqslant \frac{2K}{\pi}\int_0^\pi |f(\omega)| \sin\omega\, d\omega, \tag{$*$}$$

for some suitable constant $K > 0$.

In order to solve (5) we consider the homotopic problems

$$x_1 = \lambda H(I-P)Nx + \lambda PNx - (1-\lambda)\varepsilon(x_0 - \rho^*)\sin t, \tag{6}$$

where ε and ρ^* are defined as $\rho^* = (\rho_1+\rho_2)/2$;

$$K\left[|p_0| + \frac{2}{\pi}\int_0^\pi |p(t)|\sin t\, dt + \varepsilon(\rho_2-\rho_1)/2\right] < \delta. \tag{7}$$

Let $R_0 > 0$ be such that $R_0 > \delta$ and let Ω be defined by $\Omega = \{x(t) \in C[0,\pi] : x(t) = x_0 \sin t + x_1(t),\ \|x_1\| < R_0,\ \int_0^\pi x_1(t)\sin t\, dt = 0 \text{ and } \rho_1 < x_0 < \rho_2\}$.

When $\lambda = 0$, problem (6) has the unique solution $x_1 = 0$ and $x_0 = \rho^*$ with $x(t) = x_0\sin t + x_1(t) \in \Omega$. It remains to show that for $\lambda \in (0,1)$, (6) does not have a solution on the boundary of Ω. If $x(t)$ is a solution of (6) for some $\lambda \in (0,1)$, then

$$\lambda PNx - (1-\lambda)\varepsilon(x_0-\rho^*)\sin t = 0.$$

Thus

$$\begin{aligned}\lambda\frac{2}{\pi}\int_0^\pi g(x)\sin t\, dt &= \lambda\frac{2}{\pi}\int_0^\pi p(t)\sin t\, dt - (1-\lambda)\varepsilon(x_0-\rho^*)\\ &\leqslant \frac{2}{\pi}\int_0^\pi p(t)\sin t\, dt + \varepsilon\left(\frac{\rho_2-\rho_1}{2}\right).\end{aligned}$$

Since $x_1 = \lambda H(I-P)Nx$, we have

$$\begin{aligned}\|x_1\|_\infty &\leqslant K\left[|p_0| + \frac{2}{\pi}\int_0^\pi |p(t)|\sin t\, dt + \varepsilon\left(\frac{\rho_2-\rho_1}{2}\right)\right]\\ &< \delta, \quad \text{by (7)},\\ &< R_0.\end{aligned} \tag{8}$$

Hence (6) does not have a solution, for $\lambda \in (0,1)$ on the boundary of Ω satisfying $\|x_1\|_\infty = R_0$. Finally we show that (6) does not have a solution for $\lambda \in (0,1)$ satisfying $x_0 = \rho_1$ or ρ_2. If possible let $x = \rho_1 \sin t + x_1(t)$ be a solution of (6), $\lambda \in (0,1)$. Then $\|x_1\|_\infty < \delta$ as above and by virtue of (3) we have

$$-PNx = \left(\frac{2}{\pi}\int_0^\pi [g(x) - p]\sin t\,dt\right)\sin t \leqslant 0,$$

and this is a contradiction to

$$\lambda PNx - (1-\lambda)\varepsilon(x_0 - \rho^*)\sin t = 0$$

when $x_0 = \rho_1$. A similar argument shows that $x_0 \neq \rho_2$. Hence by the Leray-Schauder principle (6) has a solution in $\bar{\Omega}$ for $\lambda = 1$. Thus we have:

Theorem 1. *The nonlinear problem* (1) *has a solution* $x(t) = x_0 \sin t + x_1$ *satisfying* $\|x_1\|_\infty < K[|p_0| + (2/\pi)\int_0^\pi |p|\sin t\,dt]$ *and* $\rho_1 < x_0 < \rho_2$ *when hypotheses* (2), (3) *and* (4) *hold.*

Theorem 1 generalizes several of the well-known existence results in the literature for problem (1). We do not present details of these discussions because one can proceed analagously as in **[4]** where the corresponding discussions are presented for the periodic boundary value problem. However we will present some examples now which illustrate the wide range of applicability of Theorem 1.

We first remark that Theorem 1 holds even when the inequalities (2) or (3) are reversed and the proof follows in an identical manner. Thus, in the following examples we will use inequality (2) or (3) with the appropriate signs.

Example 1. Consider the nonlinear problem

$$x'' + x + e^x = p(t), \qquad x(0) = x(\pi) = 0. \tag{9}$$

The nonlinearity $g(x) = e^x$ is not bounded, and thus does not satisfy the hypothesis of **[5]**. However g satisfies (2). Clearly $\int_0^\pi p(t)\sin t\,dt > 0$ is a necessary condition for existence of solutions. Finally, choosing δ to satisfy (4) it is easy to see that (3) is satisfied if $\int_0^\pi p(t)\sin t\,dt > 0$. Thus a necessary and sufficient condition for the existence of solutions to (9) is $\int_0^\pi p(t)\sin t\,dt > 0$.

Example 2. We now consider the problem studied in **[1, 3]**:

$$x'' + x = \alpha x^- + p(t), \qquad x(0) = x(\pi) = 0,$$

where $\alpha > 0$ and $x^- = \max\{-x(t), 0\}$. Clearly $g(x) = -\alpha x^-$ satisfies $g(x) \leqslant 0$ (inequality (2) with the sign reversed). Proceeding exactly as in the previous example, it can be seen that the necessary and sufficient condition for existence of solutions is $\int_0^\pi p(t)\sin t\,dt \leqslant 0$.

Example 3. Finally we consider the problem

$$x'' + x + e^x h(x) = p(t), \qquad x(0) = x(\pi) = 0,$$

where $h\colon R \to R$ is such that:

(a) h is nonnegative, continuous and bounded;

(b) $\exists \mu > 0$ and sequences $\{a_n\}$, $\{b_n\}$ such that $\{a_n\} \to \infty$, $\lim(b_n/a_n) = \infty$ and $h(\xi) \geqslant \mu$ for $\xi \in [a_n, b_n]$;

(c) $\exists\{\xi_n\}$ such that $\xi_n \to \infty$ and $h(\xi_n) = 0$.

Then a sufficient condition for existence of solutions is $\int_0^\pi p(t)\sin t\,dt > 0$.

In this problem $g(x) = e^x h(x)$ is once again unbounded but satisfies (2). However $\liminf_{x\to\infty} g(x) = \limsup_{x\to-\infty} g(x) = 0$.

3. We conclude with two remarks concerning the operator H. It is evident that the continuity of $H(I-P): Z \to L_\infty$ is critical in the proof of the theorem. This property is not true in the case of partial differential operators and the continuity property of H needs to be investigated further. Existence results for partial differential equations analogous to (1) by related but different methods will be published elsewhere. We also remark that the extendability of these ideas to the case when g depends on x' does not seem to be immediate.

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