NONAMENABILITY AND BOREL PARADOXICAL
DECOMPOSITIONS FOR LOCALLY COMPACT GROUPS

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Abstract. We show that a locally compact group G is not amenable if and only if it
admits a Borel paradoxical decomposition.

In 1938 A. Tarski [7] proved the following remarkable theorem. Let G be a group
acting invertibly on a set X and A \subseteq X. Then there exists a positive, finitely additive,
G-invariant measure \( \mu \) on X with \( \mu(A) = 1 \) if and only if A does not admit a
paradoxical decomposition. Here, a subset B of X admits a paradoxical decomposi-
tion (p.d.) if there exists a partition \( A_1, \ldots, A_m, B_1, \ldots, B_n \) of B and elements
\( x_1, \ldots, x_m, y_1, \ldots, y_n \) of G such that both \( \{ x_i A_i : 1 \leq i \leq m \} \) and \( \{ y_i B_i : 1 \leq i \leq n \} \)
are partitions of B. (Thus, by using G-translates, we can “pack” two copies of B
into itself.) In the above circumstances it is convenient to say that \( A_i, B_i, x_i, y_i \) is a
p.d. (for B with respect to G). An immediate consequence of Tarski’s theorem is that
a (discrete) group G is not amenable if and only if G admits a p.d. This beautiful
result thus characterizes amenability directly in terms of translates of subsets of G
with no mention of invariant means or measures. Tarski’s proof uses a deep set-theo-
retic result of D. König [3]. Is there a simpler proof available?

A natural question, raised by W. R. Emerson, is the topological analogue of the
above nonamenability theorem. Let G be a locally compact group. Let us say that G
admits a Borel p.d. if there exists a p.d. as above with every \( A_i, B_i \) a Borel subset of
G. The question then is: Is it true that G is not amenable if and only if G admits a
Borel p.d.? The object of this note is to show that the answer to this question is yes.

What about a topological analogue for Tarski’s theorem? The reader is referred to

Theorem. Let G be a locally compact group. Then G is not amenable if and only if G
admits a Borel p.d.

Proof. Trivially, if G admits a Borel p.d., then G is not amenable. Conversely,
suppose that G is not amenable. Since G is the (directed) union of its \( \sigma \)-compact,
open subgroups, there exists a \( \sigma \)-compact, nonamenable, open subgroup \( H \) of G.
Suppose that the result is true for \( H \), and let \( A_i, B_i, x_i, y_i \) be a Borel p.d. for H
as above. Let \( T \) be a transversal for the right H-cosets in G. One readily checks that
\( A_i T, B_i T, x_i, y_i \) is a p.d. for G. To show that this p.d. is Borel, we need only show

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that \( AT \) is Borel in \( G \) if \( A \) is Borel in \( H \). This is obvious if \( A \) is open in \( H \), since then \( AT \) is open in \( G \), and the result for general \( A \) follows by using the monotone class lemma. (Note that if \( \{ C_n \} \) is a decreasing sequence of subsets of \( H \), then \( \bigcap_{n=1}^{\infty} (C_n T) = (\bigcap_{n=1}^{\infty} C_n) T \).

Thus \( G \) admits a Borel p.d., so we can suppose that \( G = H \)—i.e., \( G \) is \( \sigma \)-compact.

Since \( G \) is \( \sigma \)-compact, we can find a compact, normal subgroup \( K \) of \( G \) with \( G/K \) separable. Since \( K \) is amenable and \( G \) is not amenable, we have \( G/K \) not amenable. Let \( Q: G \to G/K \) be the quotient map. If there exists a Borel p.d. involving sets \( A_i, B_i \), then, by considering \( Q^{-1}(A_i), Q^{-1}(B_i) \), we see that \( G \) admits a Borel p.d. We can therefore suppose that \( G \) is separable.

Let \( G_e \) be the identity component of \( G \). Then \( G/G_e \) is totally disconnected, and so contains a compact open subgroup \( L \). Let \( \Phi: G \to G/G_e \) be the quotient map and \( H = \Phi^{-1}(L) \). Then \( H \) is an almost connected, open and closed subgroup of \( G \). There are two cases to be considered.

(i) \( H \) is not amenable. A result of Rickert [5, 6] shows that there exists a discrete subgroup \( F \) of \( H \) isomorphic to the free group \( F_2 \) on two generators. In particular, \( F \) is closed in \( H \). Now \( H \) is separable since \( G \) is, and a result of [4] yields a Borel cross section \( B \) for the right \( F \)-cosets in \( H \). Now \( F \) is, of course, not amenable, and so by Tarski's theorem, we can find a p.d. \( A'_i, B'_i, x_i, y_i \) for \( F \). Then \( A'_iB, B'_iB, x_i, y_i \) is a p.d. for \( H \), and the p.d. is Borel since \( A'_i, B'_i \) is countable and \( B \) is Borel. We then produce a Borel p.d. for \( G \) as in the second paragraph of the present proof.

(ii) \( H \) is amenable. The group \( G \) acts on the discrete space \( G/H \) in the usual way. We claim that there does not exist a \( G \)-invariant mean on \( \ell_\infty(G/H) \). (Indeed, following the usual line of argument in this context, if \( m \) were such a mean, and \( n \) was a left invariant mean on the space \( C(H) \) of bounded, continuous, complex-valued functions on \( H \), then the map \( \phi \to m(xH \to n((\phi x)|_H)) \), where \( \phi(x)(y) = \phi(xy) \) \((x, y \in G)\), is a left invariant mean on \( C(G) \), giving \( G \) amenable and, hence, a contradiction.) By Tarski's theorem we can find a p.d. \( A_i, B_i, x_i, y_i \) for \( G/H \) with respect to \( G \). Then \( A_iH, B_iH, x_i, y_i \) is a Borel p.d. for \( G \), and we are finished.

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