NONAMENABILITY AND BOREL PARADOXICAL DECOMPOSITIONS FOR LOCALLY COMPACT GROUPS

ALAN L. T. PATERSON

ABSTRACT. We show that a locally compact group $G$ is not amenable if and only if it admits a Borel paradoxical decomposition.

In 1938 A. Tarski [7] proved the following remarkable theorem. Let $G$ be a group acting invertibly on a set $X$ and $A \subseteq X$. Then there exists a positive, finitely additive, $G$-invariant measure $\mu$ on $X$ with $\mu(A) = 1$ if and only if $A$ does not admit a paradoxical decomposition. Here, a subset $B$ of $X$ admits a paradoxical decomposition (p.d.) if there exists a partition $A_1, \ldots, A_m, B_1, \ldots, B_n$ of $B$ and elements $x_1, \ldots, x_m, y_1, \ldots, y_n$ of $G$ such that both $\{x_iA_i : 1 \leq i \leq m\}$ and $\{y_iB_i : 1 \leq i \leq n\}$ are partitions of $B$. (Thus, by using $G$-translates, we can "pack" two copies of $B$ into itself.) In the above circumstances it is convenient to say that $A_i, B_i, x_i, y_i$ is a p.d. (for $B$ with respect to $G$). An immediate consequence of Tarski's theorem is that a (discrete) group $G$ is not amenable if and only if $G$ admits a p.d. This beautiful result thus characterizes amenability directly in terms of translates of subsets of $G$ with no mention of invariant means or measures. Tarski's proof uses a deep set-theoretic result of D. König [3]. Is there a simpler proof available?

A natural question, raised by W. R. Emerson, is the topological analogue of the above nonamenability theorem. Let $G$ be a locally compact group. Let us say that $G$ admits a Borel p.d. if there exists a p.d. as above with every $A_i, B_i$ a Borel subset of $G$. The question then is: Is it true that $G$ is not amenable if and only if $G$ admits a Borel p.d.? The object of this note is to show that the answer to this question is yes.

What about a topological analogue for Tarski's theorem? The reader is referred to [2] for information about amenable locally compact groups.

THEOREM. Let $G$ be a locally compact group. Then $G$ is not amenable if and only if $G$ admits a Borel p.d.

PROOF. Trivially, if $G$ admits a Borel p.d., then $G$ is not amenable. Conversely, suppose that $G$ is not amenable. Since $G$ is the (directed) union of its $\sigma$-compact, open subgroups, there exists a $\sigma$-compact, nonamenable, open subgroup $H$ of $G$. Suppose that the result is true for $H$, and let $A_i, B_i, x_i, y_i$ be a Borel p.d. for $H$ as above. Let $T$ be a transversal for the right $H$-cosets in $G$. One readily checks that $A_iT, B_iT, x_i, y_i$ is a p.d. for $G$. To show that this p.d. is Borel, we need only show

Received by the editors November 28, 1984.
1980 Mathematics Subject Classification. Primary 43A07, 22D05.
that $AT$ is Borel in $G$ if $A$ is Borel in $H$. This is obvious if $A$ is open in $H$, since then $AT$ is open in $G$, and the result for general $A$ follows by using the monotone class lemma. (Note that if $\{C_n\}$ is a decreasing sequence of subsets of $H$, then $\bigcap_{n=1}^{\infty} (C_nT) = (\bigcap_{n=1}^{\infty} C_n)T$.)

Thus $G$ admits a Borel p.d., so we can suppose that $G = H$—i.e., $G$ is $\sigma$-compact.

Since $G$ is $\sigma$-compact, we can find a compact, normal subgroup $K$ of $G$ with $G/K$ separable. Since $K$ is amenable and $G$ is not amenable, we have $G/K$ not amenable. Let $Q : G \to G/K$ be the quotient map. If there exists a Borel p.d. involving sets $A_i, B_i$, then, by considering $Q^{-1}(A_i), Q^{-1}(B_i)$, we see that $G$ admits a Borel p.d. We can therefore suppose that $G$ is separable.

Let $G_e$ be the identity component of $G$. Then $G/G_e$ is totally disconnected, and so contains a compact open subgroup $L$. Let $\Phi : G \to G/G_e$ be the quotient map and $H = \Phi^{-1}(L)$. Then $H$ is an almost connected, open and closed subgroup of $G$. There are two cases to be considered.

(i) $H$ is not amenable. A result of Rickert [5, 6] shows that there exists a discrete subgroup $F$ of $H$ isomorphic to the free group $F_2$ on two generators. In particular, $F$ is closed in $H$. Now $H$ is separable since $G$ is, and a result of [4] yields a Borel cross section $B$ for the right $F$-cosets in $H$. Now $F$ is, of course, not amenable, and so by Tarski’s theorem, we can find a p.d. $A_i', B_i', x_i, y_i$ for $F$. Then $A_i'B_i'x_i, y_i$ is a p.d. for $H$, and the p.d. is Borel since each $A_i', B_i'$ is countable and $B$ is Borel. We then produce a Borel p.d. for $G$ as in the second paragraph of the present proof.

(ii) $H$ is amenable. The group $G$ acts on the discrete space $G/H$ in the usual way. We claim that there does not exist a $G$-invariant mean on $\ell_\infty(G/H)$. (Indeed, following the usual line of argument in this context, if $m$ were such a mean, and $n$ was a left invariant mean on the space $C(H)$ of bounded, continuous, complex-valued functions on $H$, then the map $\phi \mapsto m(xH \to n((\phi x)|_H))$, where $\phi x(y) = \phi(xy)$ ($x, y \in G$), is a left invariant mean on $C(G)$, giving $G$ amenable and, hence, a contradiction.) By Tarski’s theorem we can find a p.d. $A_i, B_i, x_i, y_i$ for $G/H$ with respect to $G$. Then $A_i'H, B_i'H, x_i, y_i$ is a Borel p.d. for $G$, and we are finished.

REFERENCES

6. ..., Amenable groups and groups with the fixed point property, Trans. Amer. Math. Soc. 127 (1967), 221–232.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, CANADA N6A 5B7

Current address: Department of Mathematics, The Edward Wright Building, Dunbar Street, University of Aberdeen, Aberdeen, Scotland, U.K.