CYCLIC VECTORS FOR BACKWARD HYPONORMAL WEIGHTED SHIFTS

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Abstract. A unilateral weighted shift $T$ on a Hilbert space $H$ is an operator such that $T e_n = w_n e_{n+1}$ for some orthonormal basis $(e_n)_{n=0}^\infty$ and weight sequence $(w_n)_{n=0}^\infty$. If we assume $w_n > 0$, for all $n$, and let $\beta(n) = w_0 \cdots w_{n-1}$ for $n > 0$ and $\beta(0) = 1$, then $T$ is unitarily equivalent to $z^f$ on the weighted space $H^2(\beta)$ of formal power series $\sum_{n=0}^\infty f(n) z^n$ such that $\sum_{n=0}^\infty |f(n)|^2 [\beta(n)]^2 < \infty$. Regarding $T$ as multiplication for $z$ on the space $H^2(\beta)$, it is shown that, if $w_n \uparrow 1$ and $f$ is analytic in a neighborhood of the unit disk, then either $f$ is cyclic for $T^*$ or $f$ is contained in a finite-dimensional $T^*$-invariant subspace. This was shown—by different methods—for the unweighted shift operator by Douglas, Shields, and Shapiro [2]. It is also shown that every finite-dimensional $T^*$-invariant subspace is of the form

$$\left( (z - a_1)^{\alpha_1} \cdots (z - a_k)^{\alpha_k} H^2(\beta) \right)^{-1},$$

for some $a_1, \ldots, a_k$ in the unit disk and $n_1, \ldots, n_k$ positive integer.

1. Introduction and notation. Let $U$ be the unilateral shift on the Hardy space $H^2$. It follows from Beurling's theorem that a function $f$ is noncyclic for $U^*$ if and only if $f \in (\varphi H^2)^{-1}$ for some inner function $\varphi$. But this is not a very useful condition for determining whether a given function is cyclic for $U^*$. In [2], Douglas, Shields and Shapiro give a much more useful characterization (Theorem 2.2.1), which has as one of its consequences Theorem 2.2.4, which states that if $f$ is analytic in a neighborhood of the unit disk then $f$ is either cyclic or a rational function. Since the functions which are contained in a finite-dimensional $U^*$-invariant subspace are precisely the rational functions, Theorem 2.2.4 can be restated as follows.

**Theorem 0.** If $f$ is analytic in a neighborhood of the unit disk, then either $f$ is cyclic for $U^*$ or $f$ is contained in a finite-dimensional $U^*$-invariant subspace.

The main result of this paper is that this is true for any hyponormal weighted shift with unit norm. This is proved in §2; in §3 the finite-dimensional invariant subspaces for hyponormal weighted shifts are characterized.

Throughout this paper $T$ will be a hyponormal weighted shift with unit norm. We use the weighted space notation used in [3], and we assume that $T$ is the operator on $H^2(\beta)$ defined by $Tf = zf$. If $S$ is an operator, $\sigma(S)$, $\sigma_p(S)$, and $\sigma_{ap}(S)$ will denote
its spectrum, point spectrum, and approximate spectrum, respectively. If \( f \in H^2(\beta) \), then \([f]_*\) will be the smallest \( T^*\)-invariant subspace containing \( f \).

Since \( T \) is hyponormal with unit norm, we have \( w_n \uparrow 1 \), so \([\beta(n)]^{1/n} \to 1 \). It follows that any function in \( H^2(\beta) \) is analytic on the unit disk. If \(|\alpha| < 1\) and \( n \) is a nonnegative integer, let

\[
K_{\alpha,n} = \sum_{j=n}^{\infty} j \cdots (j-n+1) \alpha^{j-n} z^j.
\]

Then \( K_{\alpha,n} \in H^2(\beta) \) and, for any \( f \in H^2(\beta) \), we have

\[
\langle f, K_{\alpha,n} \rangle = \sum_{j=n}^{\infty} j \cdots (j-n+1) f(j) \alpha^{j-n} = f^{(n)}(\alpha).
\]

Since the function \( K_{\alpha,0} \) will be used particularly often, when it is convenient we will call it \( K_{\alpha} \).

2. The main result.

**Theorem 1.** If \( T \) is a hyponormal unilateral weighted shift with unit norm and \( f \) is analytic in a neighborhood of the unit disk, then either \( f \) is cyclic or \( f \) is contained in a finite-dimensional \( T^*\)-invariant subspace.

We prove this by way of the following lemmas.

**Lemma 1.** If \(|\alpha| < 1\) and \( f \in H^2(\beta) \), then

(i) \((T^* - \alpha)f = 0\) if and only if \( f \) is a constant multiple of \( K_{\alpha} \);

(ii) \((T^* - \alpha)\left( \frac{1}{n!} K_{\alpha,n} \right) = \frac{1}{(n-1)!} K_{\alpha,n-1} \) for any \( n > 0 \).

**Proof.**

(i) For any \( g \) in \( H^2(\beta) \), we have \( \langle g, (T^* - \alpha) K_{\alpha} \rangle = \langle (z - \alpha)g, K_{\alpha} \rangle = 0 \), so \((T^* - \alpha) K_{\alpha} = 0 \). Conversely, suppose \((T^* - \alpha)f = 0 \). Since the polynomials are dense in \( H^2(\beta) \) and \((T - \alpha)\) is bounded below (by Proposition 8.13 in [1]), if \( g \) is in \( H^2(\beta) \), then the function \((g - g(\alpha))/(z - \alpha)\) is also in \( H^2(\beta) \). Thus if \( g \in H^2(\beta) \), then

\[
\langle g, f \rangle = \left\langle \frac{g - g(\alpha)}{z - \alpha}, (T^* - \alpha) f \right\rangle + g(\alpha) H^2(\alpha) \langle 1, f \rangle
\]

\[
= \left\langle \frac{g - g(\alpha)}{z - \alpha}, (T^* - \alpha) f \right\rangle + g(\alpha) \langle 1, f \rangle = g(\alpha) \langle 1, f \rangle.
\]

Therefore \( f = \langle 1, f \rangle K_{\alpha} \).

(ii) If \( g \in H^2(\beta) \), then

\[
\langle g, (T^* - \alpha) \left( \frac{1}{n!} K_{\alpha,n} \right) \rangle = \left\langle (z - \alpha)g, \frac{1}{n!} K_{\alpha,n} \right\rangle = \frac{1}{n!} \langle (z - \alpha)g \rangle^{(n)}(\alpha)
\]

\[
= \frac{1}{(n-1)!} g^{(n-1)}(\alpha) = \langle g, \frac{1}{(n-1)!} K_{\alpha,n-1} \rangle. \quad \square
\]
Lemma 2. If $M$ is an invariant subspace for $T^*$ then

$$\sigma_{ap}(T^*|M) \cap \{|z| < 1\} = \sigma_p(T^*|M).$$

Proof. Let $\bar{\alpha} \in \sigma_{ap}(T^*|M)$ with $|\alpha| < 1$. Then there exists a sequence of functions $\{f_n\}$ in $M$ such that $\|f_n\| = 1$ and $\|(T^* - \bar{\alpha})f_n\| \to 0$. Let $f_n = c_n K_\alpha + g_n$, where $g_n \perp K_\alpha$. Then $\|(T^* - \bar{\alpha})g_n\| \to 0$. The subspace $(T^* - \bar{\alpha})\{K_\alpha\}^\bot$ is the range of $T^* - \bar{\alpha}$, which is closed by Proposition 8.13 of [1], so $T^* - \bar{\alpha} : \{K_\alpha\}^\bot \to (T^* - \bar{\alpha})\{K_\alpha\}^\bot$ is invertible. This implies $T^* - \bar{\alpha}$ is bounded below on $\{K_\alpha\}^\bot$, so $\|g_n\| \to 0$. Since $\|f_n\|$ and $\|g_n\|$ are bounded, the sequence $\{c_n\}$ is bounded, so it has a convergent subsequence $\{c_{n_k}\}$. Let $c = \lim_{k \to \infty} c_{n_k}$. Then $f_{n_k} \to c K_\alpha$, so $\bar{\alpha} \in \sigma_p(T^*|M)$. □

Lemma 3. If $f \in H^2(\beta)$ and there is a constant $C$ such that $\|q(T^*)f\| \leq C\|q\|$ for all polynomials $q$, then

$$\sigma(T^*|\{f\}_{\bullet}) \cap \{|z| < 1\} = \sigma_p(T^*|\{f\}_{\bullet}).$$

Proof. Let $\bar{\alpha} \in \sigma(T^*|\{f\}_{\bullet})$ with $|\alpha| < 1$. By Lemma 2, if $\bar{\alpha} \in \sigma_{ap}(T^*|\{f\}_{\bullet})$, then $\bar{\alpha} \in \sigma_p(T^*|\{f\}_{\bullet})$, so assume $\bar{\alpha}$ is in the compression spectrum. Then there exists a nonzero function $g$ in $\{f\}_{\bullet} \ominus (T^* - \bar{\alpha})\{f\}_{\bullet}$. Let $g^*(z) = g(\bar{z})$, and let $\{q_n\}$ be a sequence of polynomials such that $q_n \to g^*$ (in $H^2(\beta)$). Then

$$\|q_n(T^*)f - q_m(T^*)f\| \leq C\|q_n - q_m\| \to 0$$

as $n, m \to \infty$, so $\{q_n(T^*)f\}$ converges. Let $h = \lim_{n \to \infty} q_n(T^*)f$. Then $h \in \{f\}_{\bullet}$ and, for any nonnegative integer $k$, we have

$$\langle (T^* - \alpha)h, z^k \rangle = \langle h, (z - \alpha)z^k \rangle = \lim_{n \to \infty} \langle q_n(T^*)f, (z - \alpha)z^k \rangle$$

$$= \lim_{n \to \infty} \langle f, (z - \alpha)z^k q_n(\bar{z}) \rangle = \langle f, (z - \alpha)z^k g \rangle$$

$$= \langle (T^* - \bar{\alpha})T^*k f, g \rangle = 0,$$

so $(T^* - \bar{\alpha})h = 0$.

If $h = 0$, then, for any nonnegative integer $k$, we have

$$0 = \langle h, z^k \rangle = \lim_{n \to \infty} \langle q_n(T^*)f, z^k \rangle$$

$$= \lim_{n \to \infty} \langle f, q_n(\bar{z})z^k \rangle = \langle f, gz^k \rangle = \langle T^*k f, g \rangle,$$

so $g \perp \{f\}_{\bullet}$, contradicting the assumption that $g$ is a nonzero function in $\{f\}_{\bullet}$. Thus $h \neq 0$, so $\bar{\alpha} \in \alpha_p(T^*|\{f\}_{\bullet})$.

Corollary 1. If $T$ is subnormal and $f \in H^\infty(\beta)$, then

$$\sigma(T^*|\{f\}_{\bullet}) \cap \{|z| < 1\} = \alpha_p(T^*|\{f\}_{\bullet}).$$

Proof. Let $M_f$ be the operator on $H^2(\beta)$ defined by $M_f g = fg$. Then $M_f$ is bounded. Let $q$ be a polynomial and let $q^*(z) = \overline{q(\bar{z})}$. Then since $T$ is subnormal, $q^*(T)$ is also subnormal, so, in particular, it is hyponormal, so

$$\|q(T^*)f\| = \|(q^*(T))^*f\| \leq \|q^*(T)\|f\|$$

$$= \|q^*(z)f\| \leq \|M_f\||q^*|| = \|M_f\||q||.$$
Lemma 4. Let $f$ be a nonzero function in $H^2(\beta)$, such that
\[ R^2 \sum_{k=0}^{\infty} \frac{[\beta(k+n)]^4}{[\beta(k)]^2} |\hat{f}(k+n)|^2 \to 0 \]
as $n \to \infty$, for some $R > 1$. If $1/R < r < 1$, then the intersection $\sigma(T^*|f|_a) \cap \{|z| < r\}$ is nonempty.

Proof. Suppose $\sigma(T^*|f|_a) \cap \{|z| < r\} = \emptyset$. Then $(T^*|f|_a)^{-1}$ exists and $\sigma((T^*|f|_a)^{-1}) \subseteq \{|z| < 1/r\}$. Hence, \(\lim_{n \to \infty} ||(T^*|f|_a)^{-n}||^{1/n} \leq 1/r\) so, since $1/r < R$, there exists $N$ such that $||(T^*|f|_a)^{-n}||^{1/n} \leq R$, for all $n \geq N$. Thus, for $n \geq N$, we have $||(T^*|f|_a)^{-n}|| \leq R^n$. In particular,
\[ ||f|| = ||(T^*|f|_a)^{-n}T^*f|| \leq R^n||T^*f||, \]
so
\[ ||f||^2 \leq R^{2n}||T^*f||^2 = R^{2n} \sum_{k=0}^{\infty} \frac{[\beta(k+n)]^4}{[\beta(k)]^2} |\hat{f}(k+n)|^2 \to 0 \]
as $n \to \infty$, contradicting the assumption that $f$ is nonzero. $\Box$

If $f \in H^2(\beta)$ and $R > 0$, let $f_R$ be the function defined by $f_R(z) = f(Rz)$.

Lemma 5. Let $f \in H^2(\beta)$ and $R > 1$. If $f_R \in H^2(\beta)$ and $|\alpha| < 1$, then $((T^* - \alpha)f)_R \in H^2(\beta)$.

Proof. Let $h = (T^* - \alpha)f$. Then
\[ \hat{h}(n) = \hat{f}(n + 1) \frac{\beta(n+1)}{\beta(n)} - \alpha \hat{f}(n), \]
so
\[ \sum_{n=0}^{\infty} |\hat{h}(n)|^2 R^{2n} |\beta(n)|^2 \leq 2 \left( \sum_{n=0}^{\infty} [\beta(n+1)]^2 |\hat{f}(n+1)|^2 R^{2n} + |\alpha|^2 \sum_{n=0}^{\infty} [\beta(n)]^2 |\hat{f}(n)|^2 R^{2n} \right) < \infty, \]
so $h_R \in H^2(\beta)$. $\Box$

Lemma 6. If $|\alpha| < 1$ and $K_{a,n} \in [((T^* - \alpha)^m f)]$, then $K_{a,n+m} \in [f]_a$.

Proof. By induction it suffices to show that if $K_{a,n} \in [((T^* - \alpha)f)]$, then $K_{a,n+1} \in [f]_a$. If $K_{a,n} \in [((T^* - \alpha)f)]$, then $K_{a,n} \in [f]_a$, so since
\[ (1/n!)((T^* - \alpha)^n)K_{a,n} = K_a \]
(by Lemma 1), it follows that $K_a \in [f]_a$. Let $Q$ be the orthogonal projection from $[f]_a$ to $[f]_a \cap \{K_a\}^\perp$. Since $K_{a,n} \in [((T^* - \alpha)f)]$, there exists a sequence of polynomials \(\{q_k\}\) such that $q_k(T^*)(T^* - \overline{\alpha})f \to K_{a,n}$ (as $k \to \infty$). Let $f_k = Qq_k(T^*)f$. Then $f_k \in [f]_a$ and $(T^* - \alpha)f_k \to K_{a,n}$. 

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The sequence \( \{\langle 1, f_k \rangle\} \) is bounded, since \( |\langle 1, f_k \rangle| \leq \|f_k\| \) and \( T^* - \bar{\alpha} \) is bounded below on \( \{K_\alpha\}^{-1} \). Hence, it has a convergent subsequence \( \{\langle 1, f_k \rangle\} \). Let \( d = \lim_{j \to \infty} \langle 1, f_k \rangle \). If \( g \in H^2(\beta) \), then
\[
\langle g, f_k \rangle = \left\langle \frac{g - g(\alpha)}{z - \alpha}, f_k \right\rangle + \left\langle g(\alpha), f_k \right\rangle
\]
\[
= \left\langle \frac{g - g(\alpha)}{z - \alpha}, (T^* - \bar{\alpha})f_k \right\rangle + g(\alpha)\langle 1, f_k \rangle
\]
\[
\rightarrow \left( \frac{g - g(\alpha)}{z - \alpha} \right)^{(n)}(\alpha) + dg(\alpha)
\]
\[
= \frac{1}{n + 1} g^{(n+1)}(\alpha) + dg(\alpha).
\]
Thus \( f_k \to (1/(n + 1))K_{\alpha, n + 1} + dK_\alpha \) weakly as \( j \to \infty \), so \( K_{\alpha, n + 1} \in [f]_* \). □

**Proof of Theorem 1.** Let \( f \) be a function analytic in a neighborhood of the unit disk and not contained in a finite-dimensional \( T^* \)-invariant subspace. Then there exists \( R > 1 \) such that \( f_R \in H^2(\beta) \).

Let \( q \) be a polynomial with \( q(z) = \sum_{k=0}^N a_k z^k \). Then
\[
\|q(T^*)f\|^2 = \left\| \sum_{n=0}^N a_k \sum_{n=0}^\infty \frac{[\beta(k + n)]^2}{[\beta(n)]^2} \hat{f}(k + n) z^n \right\|^2
\]
\[
= \sum_{n=0}^\infty \left( \sum_{k=0}^N a_k \frac{[\beta(k + n)]^2}{[\beta(n)]^2} \hat{f}(k + n) \right)^2 [\beta(n)]^2
\]
\[
\leq \sum_{n=0}^\infty \left( \sum_{k=0}^N |a_k| \frac{[\beta(k + n)]^2}{[\beta(n)]^2} |\hat{f}(k + n)| \right)^2 [\beta(n)]^2
\]
\[
\leq \sum_{n=0}^\infty \left( \sum_{k=0}^N |a_k|^2 [\beta(k)]^2 \right)
\]
\[
\times \left( \sum_{k=0}^N |\hat{f}(k + m)|^2 \frac{[\beta(k + n)]^4}{[\beta(k)]^2[\beta(n)]^4} \right) [\beta(n)]^2
\]
\[
= \|q\|^2 \sum_{n=0}^\infty \sum_{k=0}^N |\hat{f}(k + n)|^2 \frac{[\beta(k + n)]^4}{[\beta(k)]^2[\beta(n)]^2}.
\]
So to show that there exists a constant \( C \) such that \( \|q(T^*)f\| \leq C\|q\| \) for any polynomial \( q \), it is enough to show that
\[
\sum_{n=0}^\infty \sum_{k=0}^\infty |\hat{f}(k + n)|^2 \frac{[\beta(k + n)]^4}{[\beta(k)]^2[\beta(n)]^2} < \infty.
\]

Let \( 1 < R' < R \). Then there is a constant \( C_1 \) such that \( 1/\beta(n) \leq C_1(R')^n \leq C_1(R')^{n+k} \) for all nonnegative integers \( n \) and \( k \). Then since \( [\beta(k + n)]^2/[\beta(k)]^2 \leq 1 \),
we get
\[ |\hat{f}(k + n)|^2 \frac{[\beta(k + n)]^4}{[\beta(k)]^2[\beta(n)]^2} \leq C_1^2 |\hat{f}(k + n)|^2[\beta(k + n)]^2(R')^{2(k+n)}. \]

Since \( R' < R \), there is a constant \( C_2 \) such that \((n + 1)(R')^{2n} \leq C_2 R^{2n} \), so
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |\hat{f}(k + n)|^2[\beta(k + n)]^2(R')^{2(k+n)}
\]
\[
= \sum_{n=0}^{\infty} (n + 1)|\hat{f}(n)|^2[\beta(n)]^2(R')^{2n}
\]
\[
\leq \sum_{n=0}^{\infty} |\hat{f}(n)|^2[\beta(n)]^2 C_2 R^{2n} < \infty.
\]

Fix \( 1/R < r < 1 \) and let \( 1/r < R_1 < R_2 < R \). Then for sufficiently large \( n \), we have \( |\hat{f}(n)| \leq 1/R^2 \), so for such an \( n \),
\[
R_1^{2n} \sum_{k=0}^{R_2^n} \frac{[\beta(k + n)]^4}{[\beta(k)]^2} |\hat{f}(k + n)|^2 \leq R_1^{2n} \sum_{k=0}^{R_2^n} \frac{[\beta(k + n)]^4}{[\beta(k)]^2} \left( \frac{1}{R_2^2} \right)^{2(k+n)}
\]
\[
= \left( \frac{R_1}{R_2} \right)^{2n} \sum_{k=0}^{R_2^n} \frac{[\beta(k + n)]^4}{[\beta(k)]^2} \left( \frac{1}{R_2} \right)^{2k} \leq \left( \frac{R_1}{R_2} \right)^{2n} \sum_{k=0}^{R_2^n} \left( \frac{1}{R_2} \right)^{2k} \to 0
\]
as \( n \to \infty \), so by Lemmas 3 and 4, the intersection \( \{ |z| < r \} \cap \sigma_p(T^*|[f]\ast) \) is nonempty.

Choose \( \alpha_0 \) such that \( \bar{\alpha}_0 \in \{ |z| < r \} \cap \sigma_p(T^*|[f]\ast) \). If \( \alpha_k, k = 0, \ldots, m - 1 \), are defined, let \( f_m = (T^* - \alpha_0) \cdots (T^* - \alpha_{m-1})f \). Since \( f \) is not contained in a finite-dimensional \( T^* \)-invariant subspace, it follows that \( f_m \neq 0 \). By Lemma 5 we have \((f_m)_{\ast} \in H^2(\beta) \) so, again by Lemmas 3 and 4, the intersection \( \{ |z| < r \} \cap \sigma_p(T^*|[f_m]\ast) \) is nonempty, so choose \( \alpha_m \) such that \( \bar{\alpha}_m \in \{ |z| < r \} \cap \sigma_p(T^*|[f_m]\ast) \). In this way we obtain a sequence \( \{ \alpha_k \} \) of points in the disk \( \{ |z| < r \} \).

Suppose \( \alpha \) occurs in \( \{ \alpha_k \} \) at least \( j \) times, and let \( N \) be the positive integer such that the \( j \)th occurrence of \( \alpha \) in \( \{ \alpha_k \} \) is \( \alpha_N \). Then
\[
K_{\alpha} \in \left[ f_{N-1} \right]_{\ast} = \left[ \prod_{k=0}^{N-1} (T^* - \bar{\alpha}_k)f \right]_{\ast} \subseteq \left[ (T^* - \bar{\alpha})^{j-1}f \right]_{\ast},
\]
so, by Lemma 6, the function \( K_{\alpha,j-1} \) is in \( [f]_{\ast} \), and \( g^{(j-1)}(\alpha) = 0 \) for all \( g \) in \( [f]_{\ast} \). Since this holds for any \( \alpha \) occurring in \( \{ \alpha_k \} \) and any \( j \) such that \( \alpha \) occurs at least \( j \) times in \( \{ \alpha_k \} \), any function \( g \) in \( [f]_{\ast} \) has zeros at each \( \alpha_k \), with multiplicities according to the number of occurrences in \( \{ \alpha_k \} \). Since \( \{ \alpha_k \} \) is an infinite sequence contained in \( \{ |z| < r \} \), this implies \( [f]_{\ast} = \{ 0 \} \), so \( f \) is cyclic.

3. Finite-dimensional \( T^* \)-invariant subspaces.

**Theorem 2.** Every finite-dimensional \( T^* \)-invariant subspace is of the form
\[
\left( (z - \alpha_1)^{k_1} \cdots (z - \alpha_n)^{k_n} H(\beta) \right)_{\perp}
\]
for some \( \alpha_1, \ldots, \alpha_n \) in the open unit disk and \( k_1, \ldots, k_n \) positive integers.
Proof. Let $M$ be a finite-dimensional $T^*$-invariant subspace. Then $T^*|M$ is an operator on the finite-dimensional space $M$, so it can be put in Jordan form. Thus $M$ is the direct sum of invariant subspaces $Y$ such that $T^*|Y$ has Jordan form
\[
\begin{pmatrix}
\bar{a} & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\]
for some $\alpha$ in $\mathbb{C}$, and since $\|T^*\| = 1$, we have $|\alpha| < 1$. This means that $Y$ has a basis $f_0, \ldots, f_k$ such that $(T^* - \bar{a})f_0 = 0$ and $(T^* - \bar{a})f_i = f_{i-1}$ for $i > 0$. I will show that
\[
f_i = \sum_{j=0}^{i-1} \frac{1}{j!} \langle 1, f_{i-j} \rangle K_{\alpha, j}.
\]
The proof is by induction on $i$. For any $g \in H^2(\beta)$, we have
\[
\langle g, f_0 \rangle = \left\langle \frac{g - g(\alpha)}{z - \alpha} (z - \alpha), f_0 \right\rangle + \langle g(\alpha), f_0 \rangle
\]
\[
= \left\langle \frac{g - g(\alpha)}{z - \alpha}, (T^* - \bar{a})f_0 \right\rangle + \langle 1, f_0 \rangle g(\alpha) = \langle 1, f_0 \rangle g(\alpha),
\]
so $f_0 = \langle 1, f_0 \rangle K_{\alpha, 0}$. For any $g \in H^2(\beta)$ we have
\[
\langle g, f_i \rangle = \left\langle \frac{g - g(\alpha)}{z - \alpha} (z - \alpha), f_i \right\rangle + \langle g(\alpha), f_i \rangle.
\]
The first term is
\[
\left\langle \frac{g - g(\alpha)}{z - \alpha}, (T^* - \bar{a})f_i \right\rangle = \left\langle \frac{g - g(\alpha)}{z - \alpha}, f_{i-1} \right\rangle,
\]
so if
\[
f_{i-1} = \sum_{j=0}^{i-1} \frac{1}{j!} \langle 1, f_{(i-1)-j} \rangle K_{\alpha, j},
\]
then it becomes
\[
\left\langle \frac{g - g(\alpha)}{z - \alpha}, \sum_{j=0}^{i-1} \frac{1}{j!} \langle 1, f_{(i-1)-j} \rangle K_{\alpha, j} \right\rangle
\]
\[
= \sum_{j=0}^{i-1} \frac{1}{j!} \langle 1, f_{(i-1)-j} \rangle \left( \frac{g - g(\alpha)}{z - \alpha} \right)^{(j)}(\alpha)
\]
\[
= \sum_{j=0}^{i-1} \frac{1}{j!} \langle 1, f_{(i-1)-j} \rangle g^{(j+1)}(\alpha)
\]
\[
= \sum_{j=1}^{i} \frac{1}{j!} \langle 1, f_{i-j} \rangle g^{(j)}(\alpha).
\]
Since the second term is $g(a)(l, f_i) = (1/0!)(1, f_{i-0})g^{(0)}(a)$, we get
\[
\langle g, f_i \rangle = \sum_{j=0}^{i} \frac{1}{j!} \langle 1, f_{i-j} \rangle g^{(j)}(a),
\]
so
\[
f_i = \sum_{j=0}^{i} \frac{1}{j!} \langle 1, f_{i-j} \rangle K_{a, j}.
\]

Since it is possible to solve for each $K_{a, i}$ in terms of the $f_i$'s, the set $\{ K_{a,0}, \ldots, K_{a, k} \}$ is a basis for $Y$. Since $M$ is the direct sum of spaces like $Y$, it follows that
\[
M = \left( (z - \alpha_1)^{k_1} \cdots (z - \alpha_n)^{k_n} H^2(\beta) \right) \perp
\]
for some $\alpha_1, \ldots, \alpha_n$ in the open unit disk and $k_1, \ldots, k_n$ positive integers. \(\square\)

Using Theorem 2, Theorem 1 can be restated as follows.

**Theorem 1'**. *If $f$ is analytic in a neighborhood of the unit disk and $f$ is not a linear combination of finitely many functions of the form $K_{a,n}$, where $|a| < 1$ and $n$ is a nonnegative integer, then $f$ is cyclic for $T^*$.*

**REFERENCES**


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