CYCLIC VECTORS FOR BACKWARD HYPONORMAL WEIGHTED SHIFTS

SHELLEY WALSH

Abstract. A unilateral weighted shift $T$ on a Hilbert space $H$ is an operator such that $Te_n = w_ne_{n+1}$ for some orthonormal basis $\{e_n\}_{n=0}^\infty$ and weight sequence $\{w_n\}_{n=0}^\infty$. If we assume $w_n > 0$, for all $n$, and let $\beta(n) = w_0 \cdots w_{n-1}$ for $n > 0$ and $\beta(0) = 1$, then $T$ is unitarily equivalent to $f \mapsto zf$ on the weighted space $H^2(\beta)$ of formal power series $\sum_{n=0}^\infty f(n)z^n$ such that $\sum_{n=0}^\infty |f(n)|^2[\beta(n)]^2 < \infty$. Regarding $T$ as multiplication for $z$ on the space $H^2(\beta)$, it is shown that, if $w_n \uparrow 1$ and $f$ is analytic in a neighborhood of the unit disk, then either $f$ is cyclic for $T^*$ or $f$ is contained in a finite-dimensional $T^*$-invariant subspace. This was shown—by different methods—for the unweighted shift operator by Douglas, Shields, and Shapiro [2]. It is also shown that every finite-dimensional $T^*$-invariant subspace is of the form

$$\left(\left(\sum_{i=1}^k (z - a_i)^{n_i} H^2(\beta)\right)^\perp\right),$$

for some $a_1, \ldots, a_k$ in the unit disk and $n_1, \ldots, n_k$ positive integer.

1. Introduction and notation. Let $U$ be the unilateral shift on the Hardy space $H^2$. It follows from Beurling's theorem that a function $f$ is noncyclic for $U^*$ if and only if $f \in (\varphi H^2)^\perp$ for some inner function $\varphi$. But this is not a very useful condition for determining whether a given function is cyclic for $U^*$. In [2], Douglas, Shields and Shapiro give a much more useful characterization (Theorem 2.2.1), which has as one of its consequences Theorem 2.2.4, which states that if $f$ is analytic in a neighborhood of the unit disk then $f$ is either cyclic or a rational function. Since the functions which are contained in a finite-dimensional $U^*$-invariant subspace are precisely the rational functions, Theorem 2.2.4 can be restated as follows.

**Theorem 0.** If $f$ is analytic in a neighborhood of the unit disk, then either $f$ is cyclic for $U^*$ or $f$ is contained in a finite-dimensional $U^*$-invariant subspace.

The main result of this paper is that this is true for any hyponormal weighted shift with unit norm. This is proved in §2; in §3 the finite-dimensional invariant subspaces for hyponormal weighted shifts are characterized.

Throughout this paper $T$ will be a hyponormal weighted shift with unit norm. We use the weighted space notation used in [3], and we assume that $T$ is the operator on $H^2(\beta)$ defined by $Tf = zf$. If $S$ is an operator, $\sigma(S)$, $\sigma_p(S)$, and $\sigma_{ap}(S)$ will denote
its spectrum, point spectrum, and approximate spectrum, respectively. If \( f \in H^2(\beta) \), then \( \{ f \} \) will be the smallest \( T^* \)-invariant subspace containing \( f \).

Since \( T \) is hyponormal with unit norm, we have \( w_n \uparrow 1 \), so \( [\beta(n)]^{1/n} \rightarrow 1 \). It follows that any function in \( H^2(\beta) \) is analytic on the unit disk. If \( |\alpha| < 1 \) and \( n \) is a nonnegative integer, let

\[
K_{\alpha,n} = \sum_{j=n}^{\infty} j \cdots (j - n + 1) \frac{1}{[\beta(j)]^2} \bar{\alpha}^{-n} z^j.
\]

Then \( K_{\alpha,n} \in H^2(\beta) \) and, for any \( f \in H^2(\beta) \), we have

\[
\langle f, K_{\alpha,n} \rangle = \sum_{j=n}^{\infty} j \cdots (j - n + 1) j \alpha^{-n} = f^{(n)}(\alpha).
\]

Since the function \( K_{\alpha,0} \) will be used particularly often, when it is convenient we will call it \( K_\alpha \).

2. The main result.

**Theorem 1.** If \( T \) is a hyponormal unilateral weighted shift with unit norm and \( f \) is analytic in a neighborhood of the unit disk, then either \( f \) is cyclic or \( f \) is contained in a finite-dimensional \( T^* \)-invariant subspace.

We prove this by way of the following lemmas.

**Lemma 1.** If \( |\alpha| < 1 \) and \( f \in H^2(\beta) \), then

(i) \((T^* - \alpha)f = 0 \) if and only if \( f \) is a constant multiple of \( K_\alpha \);

(ii) \((T^* - \alpha)^n K_{\alpha,n} = \frac{1}{(n-1)!} K_{\alpha,n-1} \) for any \( n > 0 \).

**Proof.** (i) For any \( g \) in \( H^2(\beta) \), we have \( \langle g, (T^* - \alpha)K_\alpha \rangle = \langle (z - \alpha)g, K_\alpha \rangle = 0 \), so \((T^* - \alpha)K_\alpha = 0 \). Conversely, suppose \((T^* - \alpha)f = 0 \). Since the polynomials are dense in \( H^2(\beta) \) and \((T - \alpha) \) is bounded below (by Proposition 8.13 in [1]), if \( g \) is in \( H^2(\beta) \), then the function \((g - g(\alpha))/(z - \alpha) \) is also in \( H^2(\beta) \). Thus if \( g \in H^2(\beta) \), then

\[
\langle g, f \rangle = \langle \frac{g - g(\alpha)}{z - \alpha}, (z - \alpha) \rangle + g(\alpha) H^2(\alpha) \langle 1, f \rangle
\]

\[
= \left( \frac{g - g(\alpha)}{z - \alpha}, (T^* - \alpha)f \right) + g(\alpha) \langle 1, f \rangle = g(\alpha) \langle 1, f \rangle.
\]

Therefore \( f = \langle 1, f \rangle K_\alpha \).

(ii) If \( g \in H^2(\beta) \), then

\[
\langle g, (T^* - \alpha)^n \rangle \left( \frac{1}{n!} K_{\alpha,n} \right) = \langle (z - \alpha)g, \frac{1}{n!} K_{\alpha,n} \rangle = \frac{1}{n!} ((z - \alpha)g)^{(n)}(\alpha)
\]

\[
= \frac{1}{(n-1)!} g^{(n-1)}(\alpha) = \langle g, \frac{1}{(n-1)!} K_{\alpha,n-1} \rangle.
\]

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Lemma 2. If $M$ is an invariant subspace for $T^*$ then
\[ \sigma_{ap}(T^*|M) \cap \{|z| < 1\} = \sigma_p(T^*|M). \]

Proof. Let $\bar{a} \in \sigma_{ap}(T^*|M)$ with $|\alpha| < 1$. Then there exists a sequence of functions \( \{f_n\} \) in $M$ such that $\|f_n\| = 1$ and $\|(T^* - \bar{a})_n\| \to 0$. Let $f_n = c_nK_\alpha + g_n$, where $g_n \perp K_\alpha$. Then $\|(T^* - \bar{a})g_n\| \to 0$. The subspace $(T^* - \bar{a})(K_\alpha)^\perp$ is the range of $T^* - \bar{a}$, which is closed by Proposition 8.13 of [1], so $T^* - \bar{a} : (K_\alpha)^\perp \to (T^* - \bar{a})(K_\alpha)^\perp$ is invertible. This implies $T^* - \bar{a}$ is bounded below on $(K_\alpha)^\perp$, so $\|g_n\| \to 0$. Since $\|f_n\|$ and $\|g_n\|$ are bounded, the sequence $\{c_n\}$ is bounded, so it has a convergent subsequence $\{c_{n_k}\}$. Let $c = \lim_{k \to \infty} c_{n_k}$. Then $f_{n_k} \to cK_\alpha$, so $\bar{a} \in \sigma_p(T^*|M)$. \( \square \)

Lemma 3. If $f \in H^2(\beta)$ and there is a constant $C$ such that $\|q(T^*)f\| \leq C\|q\|$ for all polynomials $q$, then
\[ \sigma(T^*|[f]*) \cap \{|z| < 1\} = \sigma_p(T^*|[f]*) \].

Proof. Let $\bar{a} \in \sigma(T^*|[f]*)$ with $|\alpha| < 1$. By Lemma 2, if $\bar{a} \in \sigma_{ap}(T^*|[f]*)$, then $\bar{a} \in \sigma_p(T^*|[f]*)$, so assume $\bar{a}$ is in the compression spectrum. Then there exists a nonzero function $g$ in $[f]* \ominus (T^* - \bar{a})[f]*$. Let $g^*(z) = g(\bar{z})$, and let $\{q_n\}$ be a sequence of polynomials such that $q_n \to g^*$ (in $H^2(\beta)$). Then
\[ \|q_n(T^*)f - q_m(T^*)f\| \leq C\|q_n - q_m\| \to 0 \]
as $n, m \to \infty$, so $\{q_n(T^*)f\}$ converges. Let $h = \lim_{n \to \infty} q_n(T^*)f$. Then $h \in [f]*$ and, for any nonnegative integer $k$, we have
\[ \langle (T^* - \alpha)h, z^k \rangle = \lim_{n \to \infty} \langle q_n(T^*)f, (z - \alpha)z^k \rangle = \lim_{n \to \infty} \langle f, (z - \alpha)z^k q_n(\bar{z}) \rangle = \langle f, (z - \alpha)z^k g \rangle = \langle (T^* - \bar{a})T^*k f, g \rangle = 0, \]
so $(T^* - \bar{a})h = 0$.

If $h = 0$, then, for any nonnegative integer $k$, we have
\[ 0 = \langle h, z^k \rangle = \lim_{n \to \infty} \langle q_n(T^*)f, z^k \rangle = \lim_{n \to \infty} \langle f, q_n(z)z^k \rangle = \langle f, gz^k \rangle = \langle T^*k f, g \rangle, \]
so $g \perp [f]*$, contradicting the assumption that $g$ is a nonzero function in $[f]*$. Thus $h \neq 0$, so $\bar{a} \in \sigma_p(T^*|[f]*)$.

Corollary 1. If $T$ is subnormal and $f \in H^\infty(\beta)$, then
\[ \sigma(T^*|[f]*) \cap \{|z| < 1\} = \alpha_p(T^*|[f]*) \].

Proof. Let $M_f$ be the operator on $H^2(\beta)$ defined by $M_f g = fg$. Then $M_f$ is bounded. Let $q$ be a polynomial and let $q^*(z) = \overline{q(\bar{z})}$. Then since $T$ is subnormal, $q^*(T)$ is also subnormal, so, in particular, it is hyponormal, so
\[ \|q(T^*)f\| = \|(q^*(T))^*f\| \leq \|q^*(T)f\| = \|q^*(z)f\| \leq \|M_f\|\|q^*\| = \|M_f\|\|q\|. \]
LEMMA 4. Let $f$ be a nonzero function in $H^2(\beta)$, such that
\[ R^2 n \sum_{k=0}^{\infty} \frac{[\beta(k + n)]^4}{[\beta(k)]^2} |f(k + n)|^2 \to 0 \]
as $n \to \infty$, for some $R > 1$. If $1/R < r < 1$, then the intersection $\sigma(T^*[f]*_{\omega}) \cap \{|z| < r\}$ is nonempty.

PROOF. Suppose $\sigma(T^*[f]*_{\omega}) \cap \{|z| < r\} = \emptyset$. Then $\sigma((T^*[f]*_{\omega})^{-1})$ exists and $\sigma(T^*[f]*_{\omega}) \subseteq \{|z| < 1/r\}$. Hence, $\lim_{n \to \infty} \|((T^*[f]*_{\omega})^{-1})^{1/n}\| \leq 1/r$ so, since $1/r < R$, there exists $N$ such that $\|((T^*[f]*_{\omega})^{-n})^{1/n}\| \leq R$, for all $n \geq N$. Thus, for $n \geq N$, we have $\|((T^*[f]*_{\omega})^{-n})Rn\| \leq R^n$. In particular,
\[ \|f\| = \|((T^*[f]*_{\omega})^{-n}T^nf\| \leq R^n\|T^nf\|, \]
so
\[ \|f\| \leq R^n\|T^nf\|^2 = R^n \sum_{k=0}^{\infty} \frac{[\beta(k + n)]^4}{[\beta(k)]^2} |f(k + n)|^2 \to 0 \]
as $n \to \infty$, contradicting the assumption that $f$ is nonzero. \qed

If $f \in H^2(\beta)$ and $R > 0$, let $f_R$ be the function defined by $f_R(z) = f(Rz)$.

LEMMA 5. Let $f \in H^2(\beta)$ and $R > 1$. If $f_R \in H^2(\beta)$ and $|\alpha| < 1$, then
\[ ((T^* - \alpha)f)_R \in H^2(\beta). \]

PROOF. Let $h = (T^* - \alpha)f$. Then
\[ h(n) = f(n + 1)\frac{\beta(n + 1)}{\beta(n)} - \alpha f(n), \]
so
\[ \sum_{n=0}^{\infty} |h(n)|^2 R^n |\beta(n)|^2 \leq 2 \left( \sum_{n=0}^{\infty} |\beta(n + 1)|^2 |f(n + 1)|^2 R^n \right) + |\alpha|^2 \sum_{n=0}^{\infty} |\beta(n)|^2 |f(n)|^2 R^n < \infty, \]
so $h_R \in H^2(\beta)$. \qed

LEMMA 6. If $|\alpha| < 1$ and $K_{\alpha,n} \in [(T^* - \alpha)^n]_{\omega}$, then $K_{\alpha,n+m} \in [f]_{\omega}$.

PROOF. By induction it suffices to show that if $K_{\alpha,n} \in [(T^* - \alpha)f]_{\omega}$, then $K_{\alpha,n+1} \in [f]_{\omega}$. If $K_{\alpha,n} \in [(T^* - \alpha)f]_{\omega}$, then $K_{\alpha,n} \in [f]_{\omega}$, so since
\[ (1/n!)(T^* - \alpha)^n K_{\alpha,n} = K_{\alpha} \]
(by Lemma 1), it follows that $K_{\alpha} \in [f]_{\omega}$. Let $Q$ be the orthogonal projection from $[f]_{\omega}$ to $[f]_{\omega} \cap \{K_{\alpha}\}$. Since $K_{\alpha,n} \in [(T^* - \alpha)f]_{\omega}$, there exists a sequence of polynomials $\{q_k\}$ such that $q_k(T^*)(T^* - \alpha)f \to K_{\alpha,n}$ (as $k \to \infty$). Let $f_k = Qq_k(T^*)f$. Then $f_k \in [f]_{\omega}$ and $(T^* - \alpha)f_k \to K_{\alpha,n}$. 

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The sequence \( \{ \langle 1, f_k \rangle \} \) is bounded, since \( |\langle 1, f_k \rangle| \leq \| f_k \| \) and \( T^* - \overline{a} \) is bounded below on \( \{ K_a \}^{-1} \). Hence, it has a convergent subsequence \( \{ \langle 1, f_k \rangle \} \). Let \( d = \lim_{j \to \infty} \langle 1, f_k \rangle \). If \( g \in H^2(\beta) \), then
\[
\langle g, f_k \rangle = \left\langle \frac{g - g(\alpha)}{z - \alpha}, f_k \right\rangle + \langle g(\alpha), f_k \rangle
\]
\[
= \left\langle \frac{g - g(\alpha)}{z - \alpha}, (T^* - \overline{a})f_k \right\rangle + g(\alpha)\langle 1, f_k \rangle
\]
\[
\rightarrow \left( \frac{g - g(\alpha)}{z - \alpha} \right)^{(n)}(\alpha) + dg(\alpha)
\]
\[
= \frac{1}{n + 1} g^{(n+1)}(\alpha) + dg(\alpha).
\]
Thus \( f_k \to (1/(n + 1))K_{a,n+1} + dK_a \) weakly as \( j \to \infty \), so \( K_{a,n+1} \in [f]_* \). □

PROOF OF THEOREM 1. Let \( f \) be a function analytic in a neighborhood of the unit disk and not contained in a finite-dimensional \( T^* \)-invariant subspace. Then there exists \( R > 1 \) such that \( f_R \in H^2(\beta) \).

Let \( q \) be a polynomial with \( q(z) = \sum_{k=0}^{N} a_k z^k \). Then
\[
\|q(T^*)f\|^2 = \left\| \sum_{n=0}^{N} a_k \sum_{k=0}^{\infty} \frac{[\beta(k + n)]^2}{[\beta(n)]^2} f(k + n)z^n \right\|^2
\]
\[
= \sum_{n=0}^{\infty} \left| \sum_{k=0}^{N} a_k \frac{[\beta(k + n)]^2}{[\beta(n)]^2} f(k + n) \right|^2 [\beta(n)]^2
\]
\[
\leq \sum_{n=0}^{\infty} \left( \sum_{k=0}^{N} |a_k|^2 \left[ \frac{[\beta(k + n)]^2}{[\beta(n)]^2} |f(k + n)| \right]^2 [\beta(n)]^2
\]
\[
= \left( \sum_{k=0}^{N} |f(k + n)|^2 \frac{[\beta(k + n)]^4}{[\beta(k)]^2[\beta(n)]^4} \right)[\beta(n)]^2
\]
\[
= \|q\|^2 \sum_{n=0}^{\infty} \sum_{k=0}^{N} |f(k + n)|^2 \frac{[\beta(k + n)]^4}{[\beta(k)]^2[\beta(n)]^2}.
\]
So to show that there exists a constant \( C \) such that \( \|q(T^*)f\| \leq C\|q\| \) for any polynomial \( q \), it is enough to show that
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |f(k + n)|^2 \frac{[\beta(k + n)]^4}{[\beta(k)]^2[\beta(n)]^2} < \infty.
\]
Let \( 1 < R' < R \). Then there is a constant \( C_1 \) such that \( 1/\beta(n) \leq C_1(R')^n \leq C_1(R')^{n+k} \) for all nonnegative integers \( n \) and \( k \). Then since \( [\beta(k + n)]^2/[\beta(k)]^2 \leq 1 \),
we get
\[ |\hat{f}(k + n)|^2 \frac{[\beta(k + n)]^2}{[\beta(k)]^2[\beta(n)]^2} \leq C_1^2 |\hat{f}(k + n)|^2 [\beta(k + n)]^2 (R')^{2(k+n)}. \]

Since \( R' < R \), there is a constant \( C_2 \) such that \( (n + 1)(R')^{2n} \leq C_2 R^{2n} \), so
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |\hat{f}(k + n)|^2 [\beta(k + n)]^2 (R')^{2(k+n)} = \sum_{n=0}^{\infty} (n + 1)|\hat{f}(n)|^2 [\beta(n)]^2 (R')^{2n} \\
\leq \sum_{n=0}^{\infty} |\hat{f}(n)|^2 [\beta(n)]^2 C_2 R^{2n} < \infty.
\]

Fix \( 1/R < r < 1 \) and let \( 1/r < R_1 < R_2 < R \). Then for sufficiently large \( n \), we have \( |\hat{f}(n)| \leq 1/R_2^n \), so for such an \( n \),
\[
R_1^{2n} \sum_{k=0}^{\infty} \frac{[\beta(k + n)]^4}{[\beta(k)]^2}|\hat{f}(k + n)|^2 \leq R_1^{2n} \sum_{k=0}^{\infty} \frac{[\beta(k + n)]^4}{[\beta(k)]^2} \left( \frac{1}{R_2^n} \right)^{2k+n} \\
= \left( \frac{R_1}{R_2} \right)^{2n} \sum_{k=0}^{\infty} \frac{[\beta(k + n)]^4}{[\beta(k)]^2} \left( \frac{1}{R_2^n} \right)^{2k} \leq \left( \frac{R_1}{R_2} \right)^{2n} \sum_{k=0}^{\infty} \left( \frac{1}{R_2} \right)^{2k} \to 0
\]
as \( n \to \infty \), so by Lemmas 3 and 4, the intersection \( \{|z| < r\} \cap \sigma_p(T^*[f]\downarrow) \) is nonempty.

Choose \( \alpha_0 \) such that \( \alpha_0 \in \{|z| < r\} \cap \sigma_p(T^*[f]\downarrow) \). If \( \alpha_k, k = 0, \ldots, m - 1 \), are defined, let \( f_m = (T^* - \alpha_0) \cdots (T^* - \alpha_{m-1})f \). Since \( f \) is not contained in a finite-dimensional \( T^* \)-invariant subspace, it follows that \( f_m \neq 0 \). By Lemma 5 we have \( (f_m)_m \in H^2(\beta) \) so, again by Lemmas 3 and 4, the intersection \( \{|z| < r\} \cap \sigma_p(T^*[f_m]\downarrow) \) is nonempty, so choose \( \alpha_m \) such that \( \alpha_m \in \{|z| < r\} \cap \sigma_p(T^*[f_m]\downarrow) \). In this way we obtain a sequence \( \{\alpha_k\} \) of points in the disk \( \{|z| < r\} \).

Suppose \( \alpha \) occurs in \( \{\alpha_k\} \) at least \( j \) times, and let \( N \) be the positive integer such that the \( j \)th occurrence of \( \alpha \) in \( \{\alpha_k\} \) is \( \alpha_N \). Then
\[
K_{\alpha} \in [f_{N-1}\downarrow]_* = \left[ \prod_{k=0}^{N-1} (T^* - \alpha_k)f \right]_* \subseteq [(T^* - \alpha)^{j-1}f]_*,
\]
so, by Lemma 6, the function \( K_{\alpha,j-1} \) is in \([f]_*\), and \( g^{(j-1)}(\alpha) = 0 \) for all \( g \) in \([f]_*\). Since this holds for any \( \alpha \) occurring in \( \{\alpha_k\} \) and any \( j \) such that \( \alpha \) occurs at least \( j \) times in \( \{\alpha_k\} \), any function \( g \) in \([f]_*\) has zeros at each \( \alpha_k \), with multiplicities according to the number of occurrences in \( \{\alpha_k\} \). Since \( \{\alpha_k\} \) is an infinite sequence contained in \( \{|z| < r\} \), this implies \( [f]_* = \{0\} \), so \( f \) is cyclic.

3. Finite-dimensional \( T^* \)-invariant subspaces.

**Theorem 2.** Every finite-dimensional \( T^* \)-invariant subspace is of the form
\[
\left( (z - \alpha_1)^{k_1} \cdots (z - \alpha_n)^{k_n} H(\beta) \right)_{\downarrow}
\]
for some \( \alpha_1, \ldots, \alpha_n \) in the open unit disk and \( k_1, \ldots, k_n \) positive integers.
PROOF. Let $M$ be a finite-dimensional $T^*$-invariant subspace. Then $T^*|M$ is an operator on the finite-dimensional space $M$, so it can be put in Jordan form. Thus $M$ is the direct sum of invariant subspaces $Y$ such that $T^*|Y$ has Jordan form

$$
\begin{pmatrix}
\bar{\alpha} & 0 \\
1 & 0 \\
0 & \bar{\alpha}
\end{pmatrix}
$$

for some $\alpha$ in $\mathbb{C}$, and since $\|T^*\| = 1$, we have $|\alpha| < 1$. This means that $Y$ has a basis $f_0, \ldots, f_k$ such that $(T^* - \bar{\alpha})f_0 = 0$ and $(T^* - \bar{\alpha})f_i = f_{i-1}$ for $i > 0$. I will show that

$$f_i = \sum_{j=0}^{i} \frac{1}{j!} \langle 1, f_{i-j} \rangle K_{\alpha, j}.$$

The proof is by induction on $i$. For any $g \in H^2(\beta)$, we have

$$\langle g, f_0 \rangle = \langle \frac{g - g(\alpha)}{z - \alpha} (z - \alpha), f_0 \rangle + \langle g(\alpha), f_0 \rangle$$

$$= \langle \frac{g - g(\alpha)}{z - \alpha}, (T^* - \bar{\alpha})f_0 \rangle + \langle 1, f_0 \rangle g(\alpha) = \langle 1, f_0 \rangle g(\alpha),$$

so $f_0 = \langle 1, f_0 \rangle K_{\alpha, 0}$. For any $g \in H^2(\beta)$ we have

$$\langle g, f_i \rangle = \langle \frac{g - g(\alpha)}{z - \alpha} (z - \alpha), f_i \rangle + \langle g(\alpha), f_i \rangle.$$

The first term is

$$\langle \frac{g - g(\alpha)}{z - \alpha}, (T^* - \bar{\alpha})f_i \rangle = \langle \frac{g - g(\alpha)}{z - \alpha}, f_{i-1} \rangle,$$

so if

$$f_{i-1} = \sum_{j=0}^{i-1} \frac{1}{j!} \langle 1, f_{(i-1)-j} \rangle K_{\alpha, j},$$

then it becomes

$$\langle \frac{g - g(\alpha)}{z - \alpha}, \sum_{j=0}^{i-1} \frac{1}{j!} \langle 1, f_{(i-1)-j} \rangle K_{\alpha, j} \rangle$$

$$= \sum_{j=0}^{i-1} \frac{1}{j!} \langle 1, f_{(i-1)-j} \rangle \left( \frac{g - g(\alpha)}{z - \alpha} \right)^{(j)}(\alpha)$$

$$= \sum_{j=0}^{i-1} \frac{1}{j!} \langle 1, f_{(i-1)-j} \rangle g^{(j+1)}(\alpha)$$

$$= \sum_{j=1}^{i} \frac{1}{j!} \langle 1, f_{i-j} \rangle g^{(j)}(\alpha).$$
Since the second term is $g(a)(1, f_i) = (1/0!)(1, f_{i-0}) g^{(0)}(a)$, we get

$$\langle g, f_i \rangle = \sum_{j=0}^i \frac{1}{j!} \langle 1, f_{i-j} \rangle g^{(j)}(a),$$

so

$$f_i = \sum_{j=0}^i \frac{1}{j!} \langle 1, f_{i-j} \rangle K_{a, j}.$$ 

Since it is possible to solve for each $K_{a, i}$ in terms of the $f_i$'s, the set $\{K_{a,0}, \ldots, K_{a, k}\}$ is a basis for $Y$. Since $M$ is the direct sum of spaces like $Y$, it follows that

$$M = \left((z - \alpha_1)^{k_1} \cdots (z - \alpha_n)^{k_n} H^2(\beta)\right)_\perp$$

for some $\alpha_1, \ldots, \alpha_n$ in the open unit disk and $k_1, \ldots, k_n$ positive integers. \(\Box\)

Using Theorem 2, Theorem 1 can be restated as follows.

**Theorem 1'**. If $f$ is analytic in a neighborhood of the unit disk and $f$ is not a linear combination of finitely many functions of the form $K_{a, n}$, where $|\alpha| < 1$ and $n$ is a nonnegative integer, then $f$ is cyclic for $T^*$. 

**References**


**Department of Mathematics, University of California, Berkeley, California 94720**

*Current address*: Department of Mathematics, Purdue University, West Lafayette, Indiana 47907