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A CHARACTERIZATION OF SEMISPAN OF CONTINUA

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ABSTRACT. The main results of this paper are characterizations of semispan ε, $\varepsilon \geqslant 0$, for those metric spaces which are atriodic hereditarily unicoherent continua. The results follow from Theorem 1 which gives conditions under which the union of two continua, each of semispan less than or equal to ε, has semispan less than or equal to ε.

1. Introduction. The notion of span of a compact metric space and its natural generalization semispan were defined by A. Lelek in 1964 and 1977 respectively [**6**, 7]. It follows from the definitions that semispan is greater than or equal to span and both functions are monotone with respect to closed subsets. It can be shown directly that a nonunicoherent continuum or a triod have span greater than zero. All chainable continua have semispan zero and those continua without the fixed point property have span greater than zero. W. T. Ingram constructed an example of an atriodic hereditarily unicoherent continuum with span greater than zero [**5**]. The author and J. Kell have shown that if an atriodic, hereditarily unicoherent continuum is the sum of two continua of semispan zero, then the continuum itself has semispan zero [**3**]. J. F. Davis has shown for a metric continuum in which every subcontinuum is in class W, span and semispan are equal for all subcontinua [**1**]. In particular, for continua, semispan zero and span zero are equivalent. M. Cuervo and E. Duda have shown that a hereditarily unicoherent atriodic metric continuum has span zero if and only if each indecomposable subcontinuum has span zero [**2**]. In this note we propose to demonstrate that a hereditarily unicoherent atriodic metric continuum has semispan ε, $\varepsilon \geqslant 0$, if and only if it has an indecomposable subcontinuum which has semispan ε irreducibly and has no indecomposable subcontinuum with semispan greater than ε.

2. Notations and definitions. A *continuum* is a compact connected metric space. Let π_1 and π_2 represent the natural projection mappings of the Cartesian Product $X \times X$ onto X. For a set D, represent the closure of D by $\overline{D}$, the interior of D by $\operatorname{int} D$ and the frontier or boundary of D by $\operatorname{Fr} D$. The *semispan* of a continuum (X, ρ), $\sigma_0(X)$ is the least upper bound of the set of real numbers ε such that there is a continuum Z_ε in $X \times X$ with the property that $\pi_2(Z_\varepsilon) \subset \pi_1(Z_\varepsilon)$ and $\rho(x, y) \geqslant \varepsilon$ for all (x, y) in Z_ε. The number obtained by additionally requiring $\pi_2(Z_\varepsilon) = \pi_1(Z_\varepsilon)$ is called the *span* of X and is denoted by $\sigma(X)$. A continuum is *atriodic* provided it contains no triod and is hereditarily unicoherent provided each subcontinuum is

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unicoherent. A continuum is *indecomposable* if it cannot be represented as the union of two proper subcontinua. In an atriodic hereditarily unicoherent continuum we define for each $x \in X$ the continuum $K_x = \bigcap\{K \mid K$ a subcontinuum of X and $x \in \operatorname{int} K\}$. The properties of K_x are thoroughly outlined in [2].

3. Main theorems. Suppose X is an atriodic metric continuum. In [3] it was shown that if $X = A \cup B$ and $\sigma(A) = \sigma(B) = 0$, where A, B, and $A \cap B$ are continua, then $\sigma(X) = 0$. In [4] it was shown that if $X = A \cup B$, where A and B are continua with $\sigma_0(A) \leqslant \varepsilon$, $\sigma_0(B) \leqslant \varepsilon$, $\varepsilon \geqslant 0$, X is hereditarily unicoherent, and $A \cap B$ is a continuum of continuity, then $\sigma_0(X) \leqslant \varepsilon$. The next theorem generalizes both of the results quoted above. The proof is an extension of the proof in [3] and we include a complete proof.

THEOREM 1. *If X is an atriodic hereditarily unicoherent continuum and $X = A \cup B$, where A and B are continua with $\sigma_0(A) \leqslant \varepsilon$ and $\sigma_0(B) \leqslant \varepsilon$, $\varepsilon \geqslant 0$, then $\sigma_0(X) \leqslant \varepsilon$.*

PROOF. Let $X = A \cup B$, where A and B are as in the theorem. Let $H = A \cap B$ and note that we may assume $\overline{A - B} = A$ and $\overline{B - A} = B$. If the semispan of X is greater than ε, there exists a continuum Z contained in $X \times X$ whose distance from the diagonal of $X \times X$ is greater than ε. We can make the following assumptions: (i) $\pi_2(Z) \subset \pi_1(Z)$, (ii) $\pi_1(Z)$ meets $A - B$ and $B - A$ (otherwise Z would be contained in $A \times A$ or $B \times B$, (iii) $\pi_1(Z) = X$.

We first argue that Z meets the interiors of $A \times B$ and $B \times A$. Suppose that Z misses the int $(A \times B)$. Then $\pi_1(Z \cap (A \times A)) \supset A - B$. Let $\{C_\alpha\}$ be the collection of components of $Z - (B \times X)$. Thus $\overline{C}_\alpha \cap (H \times X) \neq \varnothing$ for all α and hence $\pi_1(\overline{C}_\alpha) \cap H \neq \varnothing$ for all α. Since X is atriodic, $\pi_1(C_\alpha) \subset \pi_1(C_\beta)$ or $\pi_1(C_\beta) \subset \pi_1(C_\alpha)$ for all α and β. This and the fact that $A - B \subset \pi_1(\bigcup C_\alpha)$ imply that there exists a continuum C contained in $Z \cap (A \times A)$ with $A - B \subset \pi_1(C)$. Consequently $\pi_1(C) = A$ and this in turn implies $\sigma_0(A) > \varepsilon$. Since this is not possible, Z must meet the int$(A \times B)$ and similarly Z must meet the int$(B \times A)$.

Let $(a_1, b_1) \in \operatorname{int}(A \times B) \cap Z$ and $(b_2, a_2) \in \operatorname{int}(B \times A) \cap Z$. Further let $U_1, U_2, \ldots$ be a sequence of open sets such that: (i) $\overline{U}_{i+1} \subset U_i$ for all i, (ii) $H = \bigcap_{i=1}^{\infty} U_i$, (iii) a_1, a_2, b_1, and b_2 are not in $\overline{U}_1$. Set $E_i = A \times (B - U_i)$ and $F_i = B \times (A - U_i)$. It follows that $(a_1, b_1) \in E_i$ and $(b_2, a_2) \in F_i$ for all i. Furthermore E_i and F_i are disjoint closed sets for each i. Since Z is connected, there is a component C_i of $Z - (E_i \cup F_i)$ whose closure meets E_i and F_i. Note that $\pi_2(C_i)$ meets $A - B$ and $B - A$ for each i and hence $\pi_2(C_i) \supset H$ for all i. The remainder of the proof reduces to two cases: (i) $\pi_1(\overline{C}_i)$ meets $A - B$ and $B - A$ for all i, (ii) $\pi_1(\overline{C}_i)$ is contained in A or B for some i.

Case (i). If $\pi_1(\overline{C}_i)$ meets $A - B$ and $B - A$ for all i, then for each i there exists an integer N_i such that for $j \geqslant N_i$ there is a component L_{ij} of $\overline{C}_i \cap (U_j \times X)$ whose closure meets $(\operatorname{Fr}(U_j) \cap A) \times X$ and $(\operatorname{Fr}(U_j) \cap B) \times X$. Obviously $\pi_1(\overline{L}_{ij}) \supset H$ for each $j \geqslant N_i$. Some subsequence of $\{\overline{L}_{ij}\}$, $j \geqslant N_i$, converges to a continuum L_i contained in $\overline{(H \times X) \cap Z - (E_i \cup F_i)}$. Since $H \subset \pi_1(\overline{L}_{ij})$ for all $j \geqslant N_j$, it follows that $H \subset \pi_1(L_i)$. The sequence $\{L_i\}$ has a subsequence which converges to a continuum L contained in $(H \times H) \cap Z$ and clearly $\pi_1(L) = H$ and $\pi_2(L) \subset H$. Thus $\pi_1(L) \supset \pi_2(L)$ and this implies $\sigma_0(H) > \varepsilon$ which is not possible.

Case (ii). We can assume that $\pi_1(\overline{C}_i)$ is contained in A for some i. Thus $\overline{C}_i$ is contained in $A \times X$ and meets $H \times (A - H)$ so these three sets have a point (h, a) in common. For $j \geqslant i$ the closure of the component D_j of $\overline{C}_i - E_j$ containing (h, a) meets E_j. Hence $H \subset \pi_2(\overline{D}_j)$ for each $j \geqslant i$ and so, $H \subset \pi_2(\bigcap_{j \geqslant i} \overline{D}_j)$. Let $D = \bigcap_{j \geqslant i} \overline{D}_j$, then $(h, a) \in D$ and D is contained in $A \times A$. We can now assume $H - \pi_1(D) \neq \varnothing$ and $\pi_2(D) - H$ is a proper subset of $\pi_1(D) - H$, otherwise in either case D would have its projection in one direction contained in the projection of D in the other direction. Let $H - \pi_1(D) = \bigcup_{i=1}^{\infty} M_i$, where M_i is compact and $M_i \subset \operatorname{int} M_{i+1} \neq \varnothing$ (relative to H) for all i. Let L_i be the closure of the component of $D - (A \times M_i)$ which contains (h, a). Then $\pi_2(L_i) \cap M_i \neq \varnothing$, so $\pi_2(L_i)$ contains a point of H not in $\pi_1(D)$. If $\pi_2(L_i)$ does not contain $\pi_1(D) \cap H$, then $\pi_2(L_i) - H$ is a connected open set whose closure does not contain $\pi_1(D) \cap H$ (otherwise $\pi_1(D) \cap H \subset \overline{\pi_2(L_i) - H} \subset \pi_2(L_i)$). In this event $\overline{\pi_2(L_i) - H}$, $\pi_2(L_i) \cap H$, $\pi_1(D) \cap H$ determine a triod so it must be the case that $\pi_2(L_i) \cap H \supset \pi_1(D) \cap H$.

Let N_1 be the limit of a convergent subsequence of the L_i and notice that $\pi_2(N_1) \cap H = \pi_1(D) \cap H \supset \pi_1(N_1) \cap H$ and that $(h, a) \in N_1$. Now if $\pi_2(N_1) \cap H = \pi_1(N_1) \cap H$ or $\pi_2(N_1) - H \supset \pi_1(N_1) - H$, then N_1 is a continuum in $A \times A$ with the required projections to show $\sigma_0(A) > \varepsilon$. Thus we can assume $\pi_1(N_1) \cap H$ is a proper subcontinuum of $\pi_2(N_1) \cap H$ and $\pi_1(N_1) - H$ properly contains $\pi_2(N_1) - H$.

By taking $\pi_2(N_1) \cap H - \pi_1(N_1) \cap H = \bigcup_{i=1}^{\infty} M_i$, where M_1 is compact and $M_i \subset \operatorname{int} M_{i+1} \neq \varnothing$ (relative to $\pi_2(N_1) \cap H$), a new sequence of continua L_i can be constructed with the properties that $(h, a) \in L_i$ for all i and $\pi_2(L_i) \cap H \supset \pi_1(N_1) \cap H$. Let N_2 be a continuum which is the limit of a convergent subsequence of the L_i. Then $(h, a) \in N_2$ and $\pi_2(N_2) \cap H = \pi_1(N_1) \cap H \supset \pi_1(N_2) \cap H$. Now, as before, if $\pi_2(N_2) \cap H = \pi_1(N_2) \cap H$ or $\pi_2(N_2) - H \supset \pi_1(N_2) - H$, then N_2 is the required continuum.

We can repeat the construction above until for some m, N_m is the required continuum or we obtain an infinite sequence $\{N_m\}$ with the following properties. The point $(h, a) \in N_m$ for all m and $\pi_2(N_{m+1}) \cap H = \pi_1(N_m) \cap H$ for all m. If N is the limit of a convergent subsequence of the sequence $\{N_m\}$, then $(h, a) \in N$ and $\pi_2(N) \cap H = \pi_1(N) \cap H$ and thus N is the required continuum.

THEOREM 2. *Let X be an atriodic hereditarily unicoherent continuum with semispan ε. Then X contains an indecomposable subcontinuum I which has semispan ε irreducibly.*

PROOF. Since $\sigma_0(X) = \varepsilon$, it follows by the Brouwer Reduction theorem that X contains a subcontinuum I which has semispan ε and each proper subcontinuum of I has semispan less than ε. By Theorem 1, I must be indecomposable.

THEOREM 3. *Let X be an atriodic hereditarily unicoherent continuum. Then $\sigma_0(K_x) \leqslant \varepsilon$ for all x in X if and only if $\sigma_0(I) \leqslant \varepsilon$ for each indecomposable subcontinuum I of X.*

PROOF. Suppose $\sigma_0(K_x) \leqslant \varepsilon$ for all $x \in X$. Let I be an indecomposable subcontinuum of X and $y \in I$. Clearly $I \subset K_y$ and by the monotone property of σ_0 we have $\sigma_0(I) \leqslant \sigma_0(K_y) \leqslant \varepsilon$.

Suppose now each indecomposable subcontinuum has semispan less than or equal to ε. If $x \in X$ and $\sigma_0(K_x) > \varepsilon$, then by Theorem 1, K_x would contain an indecomposable subcontinuum I with $\sigma_0(I) > \varepsilon$. Since this is not possible, $\sigma_0(K_x) \leqslant \varepsilon$ for all $x \in X$.

COROLLARY 1. *The following are equivalent for an atriodic hereditarily unicoherent continuum*:

(i) $\sigma_0(X) \leqslant \varepsilon$;

(ii) $\sigma_0(K_x) \leqslant \varepsilon$ *for all* $x \in X$;

(iii) $\sigma_0(I) \leqslant \varepsilon$ *for all indecomposable subcontinua* I *of* X.

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