NONEXISTENCE OF SOME NONPARAMETRIC SURFACES OF PRESCRIBED MEAN CURVATURE

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ABSTRACT. If Ω ⊂ ℝ² has a reentrant corner, the Dirichlet problem for the equation of prescribed mean curvature H with zero boundary value has no solution when H has constant nonzero sign.

Suppose Ω is a bounded open subset of ℝⁿ, φ ∈ C⁰(∂Ω), and H ∈ C⁰(Ω). Does there exist a solution u ∈ C⁰(Ω) ∩ C²(Ω) of the Dirichlet problem

\[ \text{div}(Tu) = nH \quad \text{on } \Omega, \]
\[ u = \phi \quad \text{on } \partial \Omega, \]

where Tu = \nabla u/W and W = \sqrt{1 + |\nabla u|^2}? Using barrier arguments, Graham Williams [5] has proven this theorem.

Given n ≥ 2, K ∈ (0, 1/√{n-1}), R > 0, and Ω with locally Lipschitz boundary satisfying a uniform exterior sphere condition of radius R, for \|H\|_∞ small enough there exists ε > 0 depending on n, K, R, and \|H\|_∞ such that whenever φ ∈ C⁰,₁(∂Ω) has Lipschitz constant ≤ K and \sup φ - \inf φ < ε, there exists u ∈ C⁰(Ω) ∩ C²(Ω) which satisfies (1).

It has been known for some time that if the data is small and ∂Ω is smooth, (1) has a (classical) solution [4].

We will say that Ω ⊂ ℝ² has a reentrant corner at P ∈ ∂Ω if for some a, α, β with β - α > π and a > 0, \{(r, θ)|0 < r < a, α ≤ θ ≤ β\} ⊂ Ω, where (r, θ) represents polar coordinates about P. The purpose of this note is to prove the following theorem.

THEOREM. Let n = 2 and Ω be a bounded open subset of ℝ² which has reentrant corner at P ∈ ∂Ω. Suppose H ∈ C⁰(Ω) with H < 0 and φ ∈ C⁰(∂Ω) with φ(P) = 0 and φ ≥ 0. Then there is no solution u ∈ C⁰(Ω) ∩ C²(Ω) of (1).

Using the results of [1], we can determine the behavior of the variational solution of (1) near P when ∂Ω \ {P} has positive curvature. The case n = 2 is interesting because of its relationship to the "membrane analogy" of engineering, which we discuss later.

PROOF. Suppose a solution u of (1) exists. We may assume P is the origin and -3π/4 < α = β. Let V be the interior of the nonconvex quadrilateral symmetric with respect to the x-axis determined by the points (r, θ) = (0, 0), (a, β), (a, -β),...
Set \( m = \inf \{ u(r, \theta) | r = a/4, -\beta \leq \theta \leq \beta \} \). By the strict maximum principle, \( u > 0 \) on \( \Omega \) and so \( m > 0 \).

Define \( k \in C^{0}(\partial V) \) by setting \( k = 0 \) on \( \partial V \cap S \), where \( S = \{(r, \theta) | r < a/2\} \), and \( k = m \) on the two sides of \( \partial V \) not touching \( P \) and by requiring \( k \) to be linear on each of the remaining portions of \( \partial \Omega \).

Now let \( f \) be the variational solution of the Dirichlet problem for the minimal surface equation in \( V \) with boundary values \( k \). Then \( f \in C^{2}(V) \cap C^{0}(\overline{V}\setminus\{P\}) \) and \( f = k \) on \( \partial V \setminus \{P\} \). Notice \( 0 \leq f \leq m \) and \( u \geq m \) on \( r = a/4, -\beta \leq \theta \leq \beta \), so \( f \leq u \) on \( r = a/4, -\beta \leq \theta \leq \beta \). Since \( u \geq 0 \) and \( k = 0 \) on \( \partial V \cap S \), we see that \( 0 \leq f \leq u \) in \( V \cap \{(r, \theta) | r < a/4\} \). This implies \( f \in C^{0}(\overline{V}) \), in contradiction to [2, pp. 146–147]. Q.E.D.

Suppose \( \Omega \) is connected and simply connected in \( \mathbb{R}^2 \). Consider a uniform bar with cross section \( \Omega \). If we apply couples to the ends and twist the bar, then the stress function \( g \) satisfies

\[
 \Delta g = 2H \quad \text{on} \ \Omega, \quad g = 0 \quad \text{on} \ \partial \Omega,
\]

where \( H = -GA \), \( G \) is the modulus of rigidity of the bar, and \( A \) is the angle of twist per unit length of the bar. The membrane analogy is the assumption that \( g \approx u \) and \( \nabla g \approx \nabla u \) on \( \Omega \), where \( g \) solves (2) and \( u \) solves (1) with \( n = 2 \) and \( \phi = 0 \). If \( \Omega \) has a reentrant corner, no solution of (1) exists and the analogy is invalid. For a discussion of the membrane analogy and some experiments it inspired, see [3].

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References

4. R. Gulliver, personal communication.

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