PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 2, February 1986

ON A CONJECTURE OF KÀTAI CONCERNING WEAKLY COMPOSITE NUMBERS

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ABSTRACT. A number is called weakly composite if the sum of the reciprocals of its prime divisors is bounded by two. In this note it is proved that, for $n \geq n_0$, there is a weakly composite number between n and $n + \log\log\log n$.

We call a number weakly composite if the sum of the reciprocals of its distinct prime divisors is bounded by two. We prove that, for all integers $n \geq n_0$, there is a weakly composite number between n and $n + \log\log\log n$.

Let $q_1 < q_2 < \cdots < q_N$ be the prime divisors of n. We say that n is weakly composite if

$$g(n) = \sum_{j=1}^{N} \frac{1}{q_j} \leq 2. \tag{1}$$

One of the consequences of the result of Hausman [**2**] is that, for fixed k, there are infinitely many integers n such that none of $n, n+1, n+2, \ldots, n+k$ is weakly composite. In other words, if n_t, $t \geq 1$, is the sequence of weakly composite numbers then, as $t \to +\infty$,

$$\limsup(n_{t+1} - n_t) = +\infty. \tag{2}$$

On the other hand, it easily follows from the most elementary results of probabilistic number theory (see, e.g., Elliott [**1**, Chapter 5, in particular the concluding remarks]) that the average of the gaps $n_{t+1} - n_t$ is bounded, and so (2) is very slowly diverging. When lecturing at Temple University on related topics, I. Kàtai of Budapest formulated the conjecture that the gaps $n_{t+1} - n_t$ must be bounded by a function of the magnitude of $\log\log\log t$. We prove this conjecture by establishing the following result.

THEOREM. *For all sufficiently large real numbers n, there is a weakly composite number between n and $n + \log\log\log n$.*

PROOF. We use an idea of Kàtai [**3**], in which he generalizes the result of Hausman. This proof also incorporates simplifications by the referee with the referee's kind permission.

Set $k = \log\log\log n$. It suffices to show that

$$S = \sum_{n < m \leq n+k} g(m) < 2k. \tag{3}$$

Received by the editors March 28, 1985.
1980 *Mathematics Subject Classification*. Primary 10A05,10H25.

We write the left side as $S_1 + S_2$, where

$$S_1 = \sum_{n<m\le n+k} \sum_{\substack{p|m \\ p<\log n}} \frac{1}{p}$$

and

$$S_2 = \sum_{n<m\le n+k} \sum_{\substack{p|m \\ p\ge\log n}} \frac{1}{p}.$$

Now

$$S_1 = \sum_{p<\log n} \frac{1}{p} \sum_{\substack{n<m\le p+k \\ p|m}} 1 \le \sum_{p<\log n} \frac{1}{p}\left(\frac{k}{p}+1\right)$$
$$< k\sum_p p^{-2} + \sum_{p<\log n} p^{-1}.$$

From the asymptotic formula

$$\sum_{p<x} \frac{1}{p} = \log\log x + O(1)$$

and the elementary inequality

$$\sum_p p^{-2} < \sum_{p<51} p^{-2} + \frac{1}{51} < \frac{1}{2}$$

we see that

(4) $$S_1 < 3k/2 + O(1).$$

Next we treat S_2. Observe that if $h(m)$ is the number of prime divisors of m exceeding $\log n$, then

$$S_2 \le \frac{1}{\log n} \sum_{n<m\le n+k} h(m).$$

For any $m \le n+k$, we have

$$(\log n)^{h(m)} \le \prod_{p|m} p \le n+k < 2n,$$

from which, upon taking the logarithm, we see that

$$h(m) < (\log 2 + \log n)/(\log\log n).$$

Consequently, for n large, $S_2 < 2k/(\log\log n)$. Combining this with (4), we get

$$S \le k(3/2 + O(1/\log\log n)).$$

The inequality at (3) then follows for n sufficiently large. The proof is completed.

References

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2. M. Hausman, *Generalization of a theorem of Landau*, Pacific J. Math. **84** (1979), 91–95.
3. I. Kàtai, *A minimax theorem for additive functions*, Publ. Math. Debrecen **30** (1983), 249–252.

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