MIXED HADAMARD'S THEOREMS

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Dedicated to Professor Hisaharu Umegaki on his sixtieth birthday and
in celebration of his having been honoured as an emeritus
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ABSTRACT. An operator \( T \) means a bounded linear operator on a complex
Hilbert space \( H \). We give two types of mixed Hadamard's theorems containing
the terms \( T, |T|, |T^*| \) as extensions of Hadamard's theorem and mixed
Schwarz's inequality \( |(Tx, y)|^2 \leq (|T|x, x)(|T^*|y, y) \) for any \( T \) and for any \( x \)
and \( y \) in \( H \). Also we scrutinize the cases when the equalities in these mixed
Hadamard's theorems hold.

1. Statement of the results.

THEOREM 1 (MIXED HADAMARD'S TYPE 1). For any operator \( T \) on \( H \) and
any \( x_1, x_2, \ldots, x_n \) in \( H \), let \( G_n \) be defined by

\[
G_n = \begin{vmatrix}
|T|x_1, x_1 & T_1 x_1, x_2 & T_1 x_1, x_3 & \cdots & T_1 x_1, x_n \\
T_1 x_2, x_1 & |T^*|x_2, x_2 & T^*_2 x_2, x_3 & \cdots & T^*_2 x_2, x_n \\
T_1 x_3, x_1 & T^*_3 x_3, x_2 & |T^*|x_3, x_3 & \cdots & T^*_3 x_3, x_n \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
T^*_n x_n, x_1 & T^*_n x_n, x_2 & T^*_n x_n, x_3 & \cdots & |T^*|x_n, x_n
\end{vmatrix}
\]

Then

\[
0 \leq G_n \leq (|T|x_1, x_1) \prod_{j=2}^{n} (|T^*|x_j, x_j)
\]

and \( G_n = 0 \) if and only if \( S_1 = \{ T|x_1, T^*x_2, T^*x_3, \ldots, T^*x_n \} \) is a system of linearly
dependent vectors if and only if \( S_2 = \{ T|x_1, |T^*|x_2, |T^*|x_3, \ldots, |T^*|x_n \} \) is a system
of linearly dependent vectors. On the right-hand side, equality holds if and only if
\( (Tx_j, x_j) = 0 \) for \( j = 2, 3, \ldots, n \) and \( (T^*x_j, x_k) = 0 \) for \( j < k \) \((j = 2, 3, \ldots, n - 1)\)
or \( S_1 \) contains the zero vector (equivalently, \( S_2 \) contains the zero vector).

THEOREM 2 (MIXED HADAMARD'S TYPE 2). For any operator \( T \) on \( H \) and
any \( x_1, x_2, \ldots, x_n \) in \( H \), let \( G_{2n} \) be defined by

\[
G_{2n} = \begin{vmatrix}
|T|x_1, x_1 & T_1 x_1, x_2 & T_1 x_1, x_3 & \cdots & T_1 x_1, x_n \\
T_1 x_2, x_1 & |T^*|x_2, x_2 & T^*_2 x_2, x_3 & \cdots & T^*_2 x_2, x_n \\
T_1 x_3, x_1 & T^*_3 x_3, x_2 & |T^*|x_3, x_3 & \cdots & T^*_3 x_3, x_n \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
T^*_n x_n, x_1 & T^*_n x_n, x_2 & T^*_n x_n, x_3 & \cdots & |T^*|x_n, x_n
\end{vmatrix}
\]

Received by the editors November 19, 1984 and, in revised form, February 11, 1985.
1980 Mathematics Subject Classification. Primary 47A30; Secondary 47A99.
Key words and phrases. Hadamard’s theorem, Schwarz’s inequality, polar decomposition.

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0002-9939/86 $1.00 + .25 per page
Then
\[ 0 \leq G_{2n} \leq \prod_{j=1}^{2n-1} (|T|x_j,x_j)(|T^*|x_{j+1},x_{j+1}) \]
and \( G_{2n} = 0 \) if and only if \( S_1 = \{|T|x_1,T^*x_2,|T|x_3,T^*x_4,\ldots,|T|x_{2n-1},T^*x_{2n}\} \) is a system of linearly dependent vectors if and only if
\[ S_2 = \{Tx_1,|T^*|x_2,Tx_3,|T^*|x_4,\ldots,Tx_{2n-1},|T^*|x_{2n}\} \]
is a system of linearly dependent vectors. On the right-hand side, equality holds if and only if \( (|T^*|x_{2j},x_{2k}) = 0 \) for \( j \neq k \), \( (|T|x_{2j-1},x_{2k-1}) = 0 \) for \( j \neq k \), and \( (Tx_{2j-1},x_{2k}) = 0 \) for \( j,k = 1,2,\ldots,n \), or \( S_1 \) contains the zero vector (equivalently, \( S_2 \) contains the zero vector).

COROLLARY 1 (MIXED SCHWARZ’S INEQUALITY). For any operator \( T \) and any \( x,y \) in \( H \), then
\[ (Tx,y)^2 \leq (|T|x,x)(|T^*|y,y). \]
The equality holds if and only if \(|T|x \) and \( T^*y \) are linearly dependent if and only if \( Tx \) and \( T^*y \) are linearly dependent.

REMARK. We would like to emphasize that the equality holds if and only if \(|T|x \) and \( T^*y \) are linearly dependent if and only if \( Tx \) and \( T^*y \) are linearly dependent. One might believe that the equality would hold if and only if \(|T|x \) and \( |T^*|y \) are linearly dependent. But here we can give a simple counterexample as follows. Let
\[ T = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 \\ 4 \end{pmatrix}. \]
Then
\[ |T^*|y = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2|T|x, \]
that is, \(|T|x \) and \(|T^*|y \) are linearly dependent, but
\[ (Tx,y)^2 = 36 \neq (|T|x,x)(|T^*|y,y) = 54. \]
This mixed Schwarz’s inequality is discussed in [3, Problem 138] except the case when the equality holds.

2. Proofs of the results.
In order to show the results, we need the following

THEOREM A. For \( x_1,x_2,\ldots,x_n \) in \( H \), let \( G_n \) be the determinant of a square matrix of order \( n \) defined by \( G_n = |((x_j,x_k))| \). Then
\[ 0 \leq G_n \leq ||x_1||^2||x_2||^2\cdots||x_n||^2. \]
On the left-hand side, equality holds if and only if \( x_1,x_2,\ldots,x_n \) are linearly dependent. On the right-hand side, equality holds if and only if \( x_1,x_2,\ldots,x_n \) are mutually orthogonal or \( \{x_1,x_2,\ldots,x_n\} \) contains the zero vector.

The right-hand-side inequality in Theorem A is Hadamard’s theorem and also the left-hand-side inequality in Theorem A is well known and can be considered as a generalization of Schwarz’s inequality. Many ingenious and elegant proofs of Hadamard’s theorem have been given by many authors (for example [1, 2, 4, 5]).
THEOREM B. Let \( T = U|T| \) be the polar decomposition of \( T \) where \( U \) means the partial isometry and \( |T| = (T^*T)^{1/2} \) with \( N(U) = N(|T|) \) where \( N(S) \) denotes the kernel of an operator \( S \). Then

(i) \( |T^*| = U|T|U^* = UT^* \);
(ii) \( T^* = U^*|T^*| \) is also the polar decomposition of \( T^* \) with \( N(U^*) = N(|T^*|) \).

PROOF OF THEOREM B. Theorem B is well known, but for the sake of convenience, we cite the proof. (i) As \( U^*U \) is the initial projection and \( U^*U|T| = |T| \), so that \( |T^*|^2 = TT^* = U|T|T^*U^* = U|T|U^*U|T|U^* = (U|T|U^*)^2 \), then we have \( |T^*| = U|T|U^* \) because \( U|T|U^* \) is positive. Therefore \( |T^*| = U|T|U^* = UT^* \).

(ii) By (i), we have \( U^*|T^*| = U^*U|T|U^* = |T^*|U^* = T^* \) and \( U^*x = 0 \) if and only if \( UU^*x = 0 \) if and only if \( |T^*|U^*x = 0 \) by \( N(U) = N(|T|) \) if and only if \( T^*x = 0 \) if and only if \( |T^*|x = 0 \). Then \( N(U^*) = N(|T^*|) \) and \( U^* \) is also a partial isometry. So the proof of (ii) is complete.

PROOF OF THEOREM 1. In Theorem A, we replace \( x_1 \) by \( |T|^{1/2}x_1 \) and \( x_k \) by \( |T|^{1/2}U^*x_k \) for \( k = 2, 3, \ldots, n \). Then we have the following by Theorem B:

\[
\begin{align*}
& (|T|^{1/2}x_1, |T|^{1/2}U^*x_k) = (U|T|x_1, x_k) = (Tx_1, x_k) \quad \text{for} \quad k = 2, 3, \ldots, n, \\
& (|T|^{1/2}U^*x_j, |T|^{1/2}U^*x_k) = (U|T|U^*x_j, x_k) = (|T^*|x_j, x_k) \quad \text{for} \quad j, k = 2, 3, \ldots, n.
\end{align*}
\]

By Theorem A and Theorem B, we have

\[
0 \leq G_n \leq \| |T|^{1/2}x_1 \|^2 \| |T|^{1/2}U^*x_2 \|^2 \cdots \| |T|^{1/2}U^*x_n \|^2
= (|T|x_1, x_1)(|T^*|x_2, x_2) \cdots (|T^*|x_n, x_n).
\]

\( G_n = 0 \) if and only if \( |T|^{1/2}x_1, |T|^{1/2}U^*x_2, \ldots, |T|^{1/2}U^*x_n \) are linearly dependent (by Theorem A) if and only if \( |T|x_1, |T|U^*x_2, \ldots, |T|U^*x_n \) are linearly dependent (by the positivity of \( |T|^{1/2} \)) if and only if \( S_1 = \{|T|x_1, T^*x_2, T^*x_3, \ldots, T^*x_n\} \) is a system of linearly dependent vectors (by Theorem B). Then

\[
US_1 = \{U|T|x_1, UT^*x_2, UT^*x_3, \ldots, UT^*x_n\}
\]

is a system of linearly dependent vectors if and only if

\[
S_2 = \{Tx_1, T^*x_2, T^*x_3, \ldots, T^*x_n\}
\]

is a system of linearly dependent vectors (by Theorem B).

Conversely assume that \( S_2 \) is a system of linearly dependent vectors. Then \( U^*S_2 = \{U^*Tx_1, U^*T^*x_2, U^*T^*x_3, \ldots, U^*T^*x_n\} \) is a system of linearly dependent vectors if and only if \( S_1 = \{|T|x_1, T^*x_2, T^*x_3, \ldots, T^*x_n\} \) is a system of linearly dependent vectors by Theorem B, so that \( S_1 \) is a system of linearly dependent vectors if and only if \( S_2 \) is a system of linearly dependent vectors. The proof of equality for the right-hand side follows from Theorem A and the argument stated above in the first half of the proof. So the proof of Theorem 1 is complete.

PROOF OF THEOREM 2. In Theorem A we replace \( x_{2k} \) by \( |T|^{1/2}U^*x_{2k} \) for \( k = 1, 2, \ldots, n \) and \( x_{2k-1} \) by \( |T|^{1/2}x_{2k-1} \) for \( k = 1, 2, \ldots, n \). Then by Theorem B we have

\[
\begin{align*}
(|T|^{1/2}U^*x_{2j}, |T|^{1/2}U^*x_{2k}) &= (U|T|U^*x_{2j}, x_{2k}) \\
&= (|T^*|x_{2j}, x_{2k}) \quad \text{for} \quad j, k = 1, 2, \ldots, n, \\
(|T|^{1/2}x_{2j-1}, |T|^{1/2}U^*x_{2k}) &= (U|T|x_{2j-1}, x_{2k}) \\
&= (Tx_{2j-1}, x_{2k}) \quad \text{for} \quad j, k = 1, 2, \ldots, n.
\end{align*}
\]
By Theorem A and Theorem B, we have

\[ 0 \leq G_{2n} \leq \left| \| T^{1/2} x_1 \| \right|^2 \left| \| T^{1/2} U^* x_2 \| \right|^2 \cdots \left| \| T^{1/2} x_{2n-1} \| \right|^2 \left| \| T^{1/2} U^* x_{2n} \| \right|^2 \\
= (|T| x_1, x_1)(|T^*| x_2, x_2) \cdots (|T| x_{2n-1}, x_{2n-1})(|T^*| x_{2n}, x_{2n}). \]

Since the proofs of the left-hand side and the right-hand side of the equality are given in the same way as in the proofs of Theorem 1, we omit them.

**Proof of Corollary 1.** The proof follows from the inequality in Theorem 1 or Theorem 2.

**References**