LARGE-TIME BEHAVIOR OF SOLUTIONS TO CERTAIN QUASILINEAR PARABOLIC EQUATIONS IN SEVERAL SPACE DIMENSIONS

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Abstract. We consider the Cauchy problem, $u_t + \text{div} f(u) = \Delta u$ for $x \in \mathbb{R}^n$, $t > 0$ with $u(x, 0) = u_0(x)$. For $n = 1$, suppose $f'' > 0$ and $\int |u_0 - \phi| \, dx < \infty$ where $\phi$ is piecewise constant and $\phi(x) \to u^+(u^-)$ as $x \to +\infty (-\infty)$. A result of Il'in and Oleinik states that if $\phi(x - kt)$ is an entropy solution of $u_t + \text{div} f(u) = 0$, then $u(x, t)$ approaches a traveling wave solution, $\bar{u}(x - kt)$, as $t \to \infty$, with $\bar{u}(x) \to u^+(u^-)$ as $x \to +\infty (-\infty)$. We give two examples which show that this result does not hold for $n \geq 2$.

This work concerns the asymptotic behavior of the solution to the Cauchy problem,

\begin{align*}
&u_t + \text{div} f(u) = \Delta u \quad \text{for } x \in \mathbb{R}^n, \; t > 0, \\
&u(x, 0) = u_0(x) \quad \text{for } x \in \mathbb{R}^n,
\end{align*}

where $f \in C^2(\mathbb{R}; \mathbb{R}^n)$, and $n \geq 2$. We assume throughout the paper that $u_0$ is a bounded function which approaches a piecewise constant state as $|x| \to \infty$ in the sense that $\int_{\mathbb{R}^n} |u_0 - \phi| \, dx < \infty$, where

$$\phi(x) = a \cdot \chi_{\Omega}(x) + b \cdot \chi_{\mathbb{R}^n \setminus \Omega}(x)$$

with $a \geq b$, and $\Omega$ is a connected open subset of $\mathbb{R}^n$ (with piecewise smooth boundary if $n \geq 2$). We also assume without loss of generality that $f(a) = f(b) = 0$, since the transformation $\tilde{x} = x - kt$, $\tilde{t} = t$, where $k = \frac{f(a) - f(b)}{a - b}$, yields an equation of the form (1) which satisfies this condition.

For $n = 1$, a result of Il'in and Oleinik [3] states that if $f$ is strictly convex and

$$\phi(x) = \phi_c(x) \equiv \begin{cases} a, & x < c, \\
                          b, & x > c,
\end{cases}$$

for some $c \in \mathbb{R}$, then $\lim_{t \to \infty} u(x, t) = \bar{u}(x)$ exists and satisfies

\begin{align*}
\text{div} f(\bar{u}) = \Delta \bar{u} & \quad \text{in } \mathbb{R}^n, \\
\bar{u}(x) - \phi(x) & \to 0 \quad \text{as } |x| \to \infty, \text{ and} \\
\int_{\mathbb{R}^n} (\bar{u} - \phi) \, dx & = \int_{\mathbb{R}^n} (u_0 - \phi) \, dx.
\end{align*}
Note that any function \( \phi \) of the form (3) satisfies
\[
\frac{\partial u}{\partial t} + \text{div} f(u) = 0.
\]
Also since \( f \) is convex, the functions \( \{\phi_\epsilon\} \) are precisely those of the form (3) which satisfy the entropy condition (see (E) below). Thus in one space dimension, \( \lim_{t \to \infty} u(x, t) \) exists and satisfies (4) and (5) whenever \( \phi \) satisfies (E).

It is easy to see that the converse is also true. That is, if \( \lim_{t \to \infty} u(x, t) \) exists and satisfies (4) and (5), there is a function \( \psi \) of the form (3) which satisfies the entropy condition and
\[
\int_{\mathbb{R}} |\phi - \psi| \, dx < \infty.
\]

The entropy condition for solutions of (6) when \( n \geq 1 \) was formulated by Kruzkov and Vol'pert in \[4 \text{ and } 6\]. When applied to \( \phi \) as in (3), it can be stated as follows:
\[
(f(c), n(x)) \leq 0 \quad \text{for all } c \in (b, a)
\]
\( H^{n-1} \)-almost everywhere on \( \partial \Omega \), where \( n(x) \) is the outward-pointing normal to \( \Omega \) at \( x \).

In this paper we give two examples which demonstrate that Il'in and Oleinik's result on the large-time behavior of solutions to (1) and (2) fails in dimension \( n \geq 2 \). In both examples, \( f \) satisfies the strong convexity condition formulated by Conway in [1]. (In fact, we take \( f = (0, \ldots, 0, F) \) with \( F \) strictly convex.) We take \( u_0 = \phi \) where \( \phi \) has the form (3) and \( \phi \) satisfies (E).

In our first example, \( \tilde{u}(x) \equiv \lim_{t \to \infty} u(x, t) \) exists and satisfies the elliptic equation (4), but does not inherit the asymptotic values of \( \phi \) as \( |x| \to \infty \), as in (5). In our second example, \( \tilde{u} \) fails to exist.

**Example 1.** Let \( f(u) = (0, 0, \ldots, F(u)) \) where \( F(u) \) is a smooth strictly convex function with \( F(0) = F(1) = 0 \). Set \( \phi(x) = \chi_\Omega(x) \) where
\[
\Omega = \{x = (x', x_n): x_n < -|x'|\}.
\]
One readily checks that \( \phi \) satisfies condition (E).

We take \( u(x, t) \) to be the solution of (1) and (2) with \( u_0(x) = \phi(x) \). Such a solution will exist and be unique in the class
\[
C([0, \infty); L^1_{\text{loc}}(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n+1) \cap C^2(\mathbb{R}^n \times (0, \infty))
\]
and satisfies \( 0 \leq u \leq 1 \). This can be seen by considering a sequence of smooth, bounded functions \( u_0^\epsilon(x) \to \phi(x) \) in \( L^1_{\text{loc}}(\mathbb{R}^n) \). Taking the corresponding solutions of the Cauchy problem \[5, V, \text{Theorem 8.1}\] and arguing as in \[6, \text{§17.2}\], one obtains both existence and uniqueness in the indicated class. We will show that \( \lim_{t \to \infty} u(x, t) \equiv 0 \).

Consider first for the sake of comparison the function \( g_d(x_n, t) \) satisfying
\[
D_{x_n}g_d = D_t g_d + D_{x_n}(F(g_d)) \quad \text{for } -\infty < x_n < \infty, \ t > 0,
\]
\[
g_d(x_n, 0) = \begin{cases} 1, & x_n < d, \\ 0, & x_n > d. \end{cases}
\]
From [3] the functions
\[ \int_{\infty}^{y} (1 - g_d(s,t)) \, ds, \quad \int_{y}^{\infty} g_d(s,t) \, ds \]
exist, are continuous for \( y \in \mathbb{R} \) and \( t \geq 0 \), and together with
\[
(1 - g_d(-y,t)), \quad g_d(y,t)
\]
tend to zero as \( y \to +\infty \) uniformly in \( t \).

Set \( V_d(x_n,t) = \int_{x_n}^{\infty} g_d(s,t) \, ds \); this is continuous for \( t \geq 0 \) and satisfies
\[
D_{x_n} V_d = -F(-D_{x_n} V_d) + D_1 V_d, \quad V_d(x_n,0) = (d - x_n)^+.
\]
Moreover
\[
\tilde{g}_d(x_n) \equiv \lim_{t \to \infty} g_d(x_n,t) \quad \text{and} \quad \tilde{V}_d(x_n) \equiv \lim_{t \to \infty} V_d(x_n,t)
\]
both exist and are solutions to (7) and (8); they are uniquely determined respectively by
\[
\int_{d}^{\infty} \tilde{g}_d(s) \, ds + \int_{-\infty}^{d} (\tilde{g}_d(s) - 1) \, ds = 0, \quad \text{and} \quad \tilde{V}_d(x_n) = \int_{x_n}^{\infty} \tilde{g}_d(s) \, ds.
\]
Finally the convergence to \( \tilde{g}_d \) is uniform in \( x_n \).

From the maximum principle
\[
0 \leq u(x,t) \leq g_0(x_n,t) \quad \text{for} \quad x \in \mathbb{R}^n, \quad t \geq 0.
\]
Thus \( U(x,t) = \int_{x_n}^{\infty} u(x',s,t) \, ds \) is well defined and satisfies
\[
\Delta U = -F(-D_{x_n} U) + D_1 U, \quad U(x,0) = ( -|x'| - x_n)^+.
\]
For any \( d < 0 \), \( w_d = U - V_d \) satisfies
\[
\Delta w_d = a(x,t,d) D_{x_n} w_d + D_r w_d \quad \text{and} \quad w_d(x,0) \leq -d \cdot \chi_{\{|x'| \leq d\}}(x).
\]
Again using the maximum principle, \( w_d \leq h(x',t) \) for \( t \geq 0 \) where
\[
\Delta_x h = D_r h \quad \text{for} \quad x' \in \mathbb{R}^{n-1}, \quad t > 0
\]
and
\[
h(x',0) = -d \cdot \chi_{\{|x'| \leq -d\}}(x'), \quad x' \in \mathbb{R}^{n-1}.
\]
Hence
\[
\lim_{t \to \infty} U(x,t) \leq \tilde{V}_d(x_n) + \lim_{t \to \infty} h(x',t) = \tilde{V}_d(x_n).
\]
Since \( g_d(x_n,t) = g_0(x_n - d,t) \), \( \tilde{V}_d(x_n) \to 0 \) as \( d \to -\infty \). From parabolic estimates it follows that
\[
u(x,t) = -D_{x_n} U(x,t) \to 0 \quad \text{as} \quad t \to \infty.
\]

**Example 2.** Consider \( f(u) \) as above and \( u_0(x) = \phi(x) = \chi_{\Omega}(x) \) where
\[
\Omega = \{(x',x_n): x_n < \psi(x')\} \quad \text{with} \quad \psi \in C^1(\mathbb{R}^{n-1}), \quad -1 < \psi < 0,
\]
and
\[
\lim_{r \to \infty} r^{-n} \int_{|x'| < r} \psi \, dx' = \lim_{r \to \infty} \int_{|x'| < r} \psi \, dx'.
\]
Again one can check that (E) is satisfied. We show that \( \lim_{t \to \infty} u(x, t) \) cannot exist pointwise almost everywhere on \( \mathbb{R}^n \).

Using the previous remarks on \( g_d \) we see that
\[
g_1(x_n, t) \leq u(x, t) \leq g_0(x_n, t)
\]
and that the function
\[
U(x', t) = \int_{-\infty}^{0} (1 - u(x', s, t)) \, ds - \int_{0}^{\infty} u(x', s, t) \, ds
\]
is well defined and satisfies
\[
\Delta_x U = D_t U \quad \text{for} \; t > 0, \; x' \in \mathbb{R}^{n-1},
\]
\[
U(x', 0) = -\psi(x') \quad \text{for} \; x' \in \mathbb{R}^{n-1}.
\]

If \( \lim_{t \to -\infty} u(x, t) \) is well defined, then \( \lim_{t \to -\infty} U(x', t) = \bar{U}(x') \) exists. Hence \( \bar{U} \) is a bounded harmonic function and thus a constant. By a result of [2] this is true iff
\[
\lim_{r \to \infty} r^{1-n} \int_{|x'| \leq r} \psi \, dx' \quad \text{exists}.
\]

REFERENCES