PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 2, February 1986

ON THE PUNCTUAL AND LOCAL ERGODIC THEOREM FOR NONPOSITIVE POWER BOUNDED OPERATORS IN $L^p_{\mathbf{C}}[0,1]$, $1 < p < +\infty$

I. ASSANI

ABSTRACT. We show in this note that there exists a function $f \in \bigcap_{1<p<+\infty} L^p_{\mathbf{C}}[0,1]$ and for each p an isomorphism $T\colon L^p_{\mathbf{C}} \to L^p_{\mathbf{C}}$ such that $\sup_{n\in\mathbf{Z}}\|T^n\| < +\infty$ and T does not satisfy the punctual ergodic theorem.

We give also an example of a one-parameter semigroup $(T_t, t \geqslant 0)$ of power bounded operators in each $L^p_{\mathbf{C}}$ $(1 < p < +\infty)$ for which the assertion of the local ergodic theorem ($(1/t)\int_0^t T_s f\,ds$ converge almost everywhere as $t \to 0_+$ for all $f \in L^p$) fails to be true.

1. Introduction, definitions and notations. Let $(\Omega, \mathscr{A}, P)$ be a σ finite measure space and $L^p(\Omega, \mathscr{A}, P)$ $(1 < p < +\infty)$ the Banach space of complex valued measurable functions on Ω such that $\|f\|^p = \int |f(x)|^p\,d\mu < +\infty$. An operator T: $L^p(\Omega, \mathscr{A}, P) \to L^p(\Omega, \mathscr{A}, P)$ satisfies the punctual ergodic theorem (p.e.t.) if for each $f \in L^p(\Omega, \mathscr{A}, P)$

$$M_n(T)f = \frac{1}{n}\sum_{i=0}^{n-1} T^i f$$

is a.e. convergent in Ω.

It is known that T satisfies the punctual ergodic theorem in the following cases: for $1 < p < \infty$, $p \neq 2$ and T an isometry (for an invertible isometry, A. Ionescu Tulcea [**1**] and for a general isometry R. V. Chacon and S. A. McGrath [**2**]), for $1 < p < +\infty$ and T a positive contraction (M. A. Akcoglu [**3**]) and for $1 < p < \infty$ and T an isomorphism such T and T^{-1} are both power bounded and positive (A. de la Torre [**4**]). D. L. Burkholder [**5**] constructed a contraction T: $L^2 \to L^2$ which does not satisfy the punctual ergodic theorem. Akcoglu and Sucheston [**6**] remarked that T can be geometrically dilated to an isometry on an L^2 space which does not satisfy the punctual ergodic theorem.

M. Feder [**7**] constructed for each $1 < p \leqslant 2$ an isomorphism T: $L^p[0,2] \to L^p[0,2]$, so that T and T^{-1} both are power bounded, and T does not satisfy the p.e.t. In [**14**] we have shown that for each $1 < p < +\infty$ there exists a power bounded operator T: $L^p_{\mathbf{R}}[0,1] \to L^p_{\mathbf{R}}[0,1]$ such that T and T^* do not satisfy the p.e.t.

In this paper we show first that there exists for each p, $1 < p < +\infty$, an isomorphism T such that *T and T^{-1} are both power bounded* and T does not satisfy

Received by the editors November 11, 1984.
1980 *Mathematics Subject Classification*. Primary 47A35; Secondary 46E30.

the p.e.t. This result is then extended to some sequences of operators defined by regular sequences (α_{nK}), introduced in [**13**].

DEFINITION 1. Let (α_{nK}) be a sequence of positive numbers. (α_{nK}) is said to be regular if

(i) $\forall n \in \mathbf{N}, \sum_{K=0}^{\infty} \alpha_{nK} = 1$,

(ii) $\forall z \in \mathbf{C}, |z| \leqslant 1, z \neq 1, |\sum_{K=0}^{\infty} \alpha_{nK} z^K| \underset{n}{\to} 0$,

(iii) $\forall n \in \mathbf{N}$, the radius of convergence r_n of the series $\sum_{K=0}^{\infty} \alpha_{nK} z^K$ satisfies the condition $r_n > 1$.

Let T_t $(t \geqslant 0)$ be a strongly continuous one-parameter semigroup of bounded linear operators on $L_{\mathbf{C}}^p$. This means that

(i) T_t is a bounded linear operator on L^p for any $t \geqslant 0$,

(ii) $T_{t+s} f = T_t \circ T_s f$ for any $t, s \geqslant 0$ and $f \in L_{\mathbf{C}}^p$,

(iii) $\lim_{t \to 0_+} \|T_t - T_s f\| = 0$ for any $f \in L_{\mathbf{C}}^p$.

We note $S_t f = \int_0^t T_s f\, ds$. Local ergodic theorems assert that $\lim_{t \to 0_+} (S_t / t) f$ exists a.e. for all $f \in L^p$. Local ergodic theorems have been studied by many authors (see [**8**]). We shall adapt the construction given by Akcoglu and Krengel in [**9**] to get an example of local ergodic divergence for power bounded operators in L^p for each $p, 1 < p < +\infty$.

II. On the punctual ergodic theorem in $L_{\mathbf{C}}^p[0,1]$, $1 < p < +\infty$. Let (ϕ_n^K) be Haar's system. The functions ϕ_n are defined by the equations

$$\phi_1(t) = \phi_0^{(0)}(t) = 1 \quad \text{for } t \in [0,1]$$

and, for $m = 2^n + k$ with $1 \leqslant k \leqslant 2^n$, $n = 0, 1, \ldots,$

$$\phi_m(t) = \phi_n^{(K)}(t) = \begin{cases} +\sqrt{2^n} & \text{for } t \in \left[\dfrac{2K-2}{2^{n-1}}, \dfrac{2K-1}{2^{n+1}}\right), \\ -\sqrt{2^n} & \text{for } t \in \left(\dfrac{2K-1}{2^{n+1}}, \dfrac{2K}{2^{n+1}}\right], \\ 0 & \text{elsewhere.} \end{cases}$$

P. L. Ulyanov [**10**] has obtained the following result.

THEOREM 1. *There exists a function* $f \in \bigcap_{1 < p < +\infty} L^p[0,1]$, $f = \sum_{i=1}^{\infty} a_i \phi_i$, *and a permutation of the integers* π *such that the series* $\sum a_{\pi(i)} \phi_{\pi(i)}(t)$ *diverges unboundedly on* $[0,1]$ *almost everywhere.*

The first proof given in [**10**] uses a quite difficult modification of Zahorski's construction. In a recent publication [**11**] P. L. Ulyanov gives a simpler proof (in L^2). From this article and the arguments of the first paper we can get a simpler proof of this theorem. Olevskii [**15**] gave also (in L^2) another proof of this result. From this theorem we can get the theorem announced.

THEOREM 2. *There exists a function* $f \in \bigcap_{1 < p < +\infty} L^p[0,1]$ *and for each* p, $1 < p < +\infty$, *an isomorphism* T *such that* T *and* T^{-1} *are both power bounded and*

$$M_n(T) f = \frac{I + T + T^2 + \cdots + T^{n-1}}{n} f$$

does not converge almost surely.

PROOF. Let p be fixed, $1 < p < +\infty$. From Haar's system $(\phi_n^{(K)}) = (\phi_m)$ we can get an unconditional base of $L_{\mathbf{C}}^p[0,1]$, $(\tilde{\phi}_n^K) = (\tilde{\phi}_m)$. Consider the permutation π and f of the previous theorem; $(\tilde{\phi}_{\pi(m)})$ is again an unconditional base of $L_{\mathbf{C}}^p[0,1]$ and if $f = \sum a_i\tilde{\phi}_i$ then $f = \sum a_{\pi(i)}\tilde{\phi}_{\pi(i)}$.

By using Remark 1 in [7] we can get an isomorphism T (T and T^{-1} are both power bounded) and a sequence $n_1 < n_2 < \cdots < n_K < \cdots$ such that $\|M_{n_K}(T) - Q_K\| < +\infty$, where Q_K is the projection

$$Q_K\left(\sum_{i=1}^{\infty} b_i\tilde{\phi}_{\pi(i)}\right) = \sum_{i\geqslant K} b_i\tilde{\phi}_{\pi(i)}.$$

We can then deduce easily that the sequence $M_{n_K}(T)f$ does not converge almost surely on $[0,1]$. This result extends easily to the case of operators defined by a regular sequence (α_{nK}) (see Definition 1).

THEOREM 3. *Let (α_{nK}) be a regular sequence and consider the sequence of operators*

$$R_n(T) = \sum_{K=0}^{\infty} \alpha_{nK}T^K.$$

Then for each p, $1 < p < +\infty$, there exists an isomorphism T: $L^p \to L^p$ and $f \in L^p$ such that the sequence $R_n(T)f$ does not converge almost surely.

PROOF. Consider the unconditional base $(\tilde{\phi}_{\pi(m)})$ of the previous proof and let P_m be the projection on $\phi_{\pi(m)}$. As in [12] the operator T will be diagonal, i.e., $T = \sum_{i=1}^{\infty}\lambda_i P_i$, where the λ_i are complex numbers with $|\lambda_i| = 1$ and $\lambda_i \neq 1$, $\forall i$. Take λ_1, $|\lambda_1| = 1$ and $\lambda_1 \neq 1$. Choose (ε_i) such that $\varepsilon_i > 0$ and $\sum\varepsilon_i = 1$. There exist an integer n_1 and a real number δ_1 such that $|R_{n_1}(\lambda_1)| < \varepsilon_1$ and $|R_{n_1}(z) - 1| < \varepsilon_1$ if $|z - 1| < \delta_1$, by the regular properties of the sequence (α_{nK}).

If λ_i, n_i, δ_i have been chosen for $1 \leqslant i \leqslant j$, then we choose λ_{j+1} such that $|\lambda_{j+1} - 1| < \delta_i$ for $1 \leqslant i \leqslant j$. Then we take $n_{j+1} > n_j$ such that $|R_{n_{j+1}}(\lambda_i)| < \varepsilon_{j+1}$ for $1 \leqslant i \leqslant j + 1$ and $\delta_{j+1} > 0$ such that $|R_{n_{j+1}}(\lambda) - 1| < \varepsilon_{j+1}$ when $|\lambda - 1| < \delta_{j+1}$.

The end of the proof is then the same as for the previous theorem.

REMARK 4. (1) Theorem 2 shows that the Abel means of an isomorphism T such that T and T^{-1} are power bounded in L^p are not always convergent a.e. (take $\alpha_{nK} = (1/n)(1 - 1/n)^K$).

(2) An analog proof had been given in the case of contractions of L^2 in [13].

III. On the local ergodic theorem in $L_{\mathbf{C}}^p[0,1]$, $1 < p < +\infty$. Let $(\tilde{\phi}_{\pi(m)})$ be the unconditional base of the previous section, $V_k = I - Q_K$, and p fixed, $1 < p < +\infty$. We have the following lemma, containing ideas from [9].

LEMMA 5. *There exist sequences (λ_n), $n = 1,\ldots,(t_K)$, $K = 1,\ldots,$ with $\lambda_n > 0$ and $0 < t_K \to 0$ such that $\sum\|V_K - (1/t_K)S_{t_K}\| < +\infty$.*

PROOF. Choose (ε_K) $\varepsilon_K > 0$ such that $\sum\varepsilon_K < +\infty$. Then take $\lambda_1 > 0$ arbitrarily and choose $0 < t_1 < 1$ so small that

$$\left|\frac{e^{i\lambda_1 t_1} - 1}{i\lambda_1 t_1} - 1\right| < \varepsilon_1.$$

If $\lambda_1, \lambda_2, \ldots, \lambda_K$ and $t_1, \ldots, t_K$ are already chosen, choose $\lambda_{K+1} > 0$ sufficiently large so that

$$\left|\frac{e^{i\lambda_{K+1}t_m} - 1}{i\lambda_{K+1}t_m}\right| < \varepsilon_m \quad \text{for each } m = 1, 2, \ldots, K.$$

Then choose $s < t_{K+1} < 1/(K+1)$ so small that

$$\left|\frac{e^{i\lambda_n t_{K+1}} - 1}{i\lambda_n t_{K+1}} - 1\right| < \varepsilon_{K+1}.$$

THEOREM 6. *Given p, $1 < p < +\infty$, there exists a function $f \in \bigcap_{1<p<+\infty} L^p_{\mathbf{C}}[0,1]$ and a one-parameter semigroup $(T_t, t \geqslant 0)$ of power bounded operators in $L^p_{\mathbf{C}}[0,1]$ for which $(1/t)\int_0^t T_s f\,ds$ does not converge a.e. as $t \to 0_+$.*

PROOF. Let p fixed, $1 < p < +\infty$. Consider the one-parameter semigroup T_t defined by

$$T_t\left(\sum_{n=1}^{\infty} \alpha_n \tilde{\phi}_{\pi(n)}\right) = \sum_{n=1}^{\infty} e^{i\lambda_n t}\alpha_n \tilde{\phi}_{\pi(n)}$$

where the λ_n are those of the previous lemma and π the permutation in Theorem 1. Then we have $T_0 = I$ (identity),

(i) for each t, s, $T_t \circ T_s = T_{t+s}$,

(ii) $\lim_{s \to t} \|T_s f - T_t f\|_p = 0$ $\forall t \geqslant 0$ and $\forall f \in L^p_{\mathbf{C}}$,

(iii) $S_t f = \sum_{n=1}^{\infty} (1/i\lambda_n)(\exp(-i\lambda_n t) - 1)\alpha_n \phi_{\pi(n)}$ $(t > 0)$.

It is now easy to see that $(S_K/t_K)f$ does not converge almost everywhere for the function f of Theorem 1.

REMARK 7. It is known that for $T_0 = I$ and any semigroup of positive bounded linear operators on L^p with $P(\Omega) < +\infty$ we can get the local ergodic theorem (see **[8]**).

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UNIVERSITÉ PARIS VI, LABORATOIRE DE PROBABILITÉS, TOUR 56, 4 PLACE JUSSIEU, 75230 PARIS CEDEX 05, FRANCE