ON THE PUNCTUAL AND LOCAL ERGODIC THEOREM
FOR NONPOSITIVE POWER BOUNDED OPERATORS
IN $L_p^p[0, 1], 1 < p < +\infty$

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Abstract. We show in this note that there exists a function $f \in \cap_{1 < p < +\infty} L_p^p[0, 1]$ and for each $p$ an isomorphism $T: L_p^p \to L_p^p$ such that $\sup_{n \in \mathbb{Z}} \|T^n\| < +\infty$ and $T$ does not satisfy the punctual ergodic theorem.

We give also an example of a one-parameter semigroup $(T_t, t \geq 0)$ of power bounded operators in each $L_p^p (1 < p < +\infty)$ for which the assertion of the local ergodic theorem ($(1/t) \int_0^t T_s f ds$ converge almost everywhere as $t \to 0^+$ for all $f \in L_p^p$) fails to be true.

1. Introduction, definitions and notations. Let $(\Omega, \mathcal{F}, P)$ be a $\sigma$ finite measure space and $L_p^p(\Omega, \mathcal{F}, P) (1 < p < +\infty)$ the Banach space of complex valued measurable functions on $\Omega$ such that $\|f\|^p = \int |f(x)|^p d\mu < +\infty$. An operator $T: L_p^p(\Omega, \mathcal{F}, P) \to L_p^p(\Omega, \mathcal{F}, P)$ satisfies the punctual ergodic theorem (p.e.t.) if for each $f \in L_p^p(\Omega, \mathcal{F}, P)$

$$M_n(T)f = \frac{1}{n} \sum_{i=0}^{n-1} T^i f$$

is a.e. convergent in $\Omega$.

It is known that $T$ satisfies the punctual ergodic theorem in the following cases: for $1 < p < +\infty$, $p \neq 2$ and $T$ an isometry (for an invertible isometry, A. Ionescu Tulcea [1] and for a general isometry R. V. Chacon and S. A. McGrath [2]), for $1 < p < +\infty$ and $T$ a positive contraction (M. A. Akcoglu [3]) and for $1 < p < +\infty$ and $T$ an isomorphism such $T$ and $T^{-1}$ are both power bounded and positive (A. de la Torre [4]). D. L. Burkholder [5] constructed a contraction $T: L^2 \to L^2$ which does not satisfy the punctual ergodic theorem. Akcoglu and Sucheston [6] remarked that $T$ can be geometrically dilated to an isometry on an $L^2$ space which does not satisfy the punctual ergodic theorem.

M. Feder [7] constructed for each $1 < p \leq 2$ an isomorphism $T: L_p^p[0, 2] \to L_p^p[0, 2]$, so that $T$ and $T^{-1}$ both are power bounded, and $T$ does not satisfy the p.e.t. In [14] we have shown that for each $1 < p < +\infty$ there exists a power bounded operator $T: L_p^p[0, 1] \to L_p^p[0, 1]$ such that $T$ and $T^*$ do not satisfy the p.e.t.

In this paper we show first that there exists for each $p, 1 < p < +\infty$, an isomorphism $T$ such that $T$ and $T^{-1}$ are both power bounded and $T$ does not satisfy...
the p.e.t. This result is then extended to some sequences of operators defined by regular sequences \((\alpha_{nK})\), introduced in [13].

**Definition 1.** Let \((\alpha_{nK})\) be a sequence of positive numbers. \((\alpha_{nK})\) is said to be regular if

(i) \(\forall n \in \mathbb{N}, \sum_{K=0}^{\infty} \alpha_{nK} = 1\),

(ii) \(\forall z \in \mathbb{C}, |z| \leq 1, z \neq 1, |\sum_{K=0}^{\infty} \alpha_{nK} z^K| \to 0\),

(iii) \(\forall n \in \mathbb{N}\), the radius of convergence \(r_n\) of the series \(\sum_{K=0}^{\infty} \alpha_{nK} z^K\) satisfies the condition \(r_n > 1\).

Let \(T_t (t \geq 0)\) be a strongly continuous one-parameter semigroup of bounded linear operators on \(L^p\). This means that

(i) \(T_t\) is a bounded linear operator on \(L^p\) for any \(t \geq 0\),

(ii) \(T_{t+s} = T_t \circ T_s\) for any \(t, s \geq 0\) and \(f \in L^p\),

(iii) \(\lim_{t \to 0} \|T_t - T_s\| = 0\) for any \(f \in L^p\).

We note \(S_t f = \int_0^t T_t f \, ds\). Local ergodic theorems assert that \(\lim_{t \to 0} (S_t / t)f\) exists a.e. for all \(f \in L^p\). Local ergodic theorems have been studied by many authors (see [8]). We shall adapt the construction given by Akcoglu and Krengel in [9] to get an example of local ergodic divergence for power bounded operators in \(L^p\) for each \(p, 1 < p < +\infty\).

II. On the punctual ergodic theorem in \(L^p[0,1], 1 < p < +\infty\). Let \((\phi^K_n)\) be Haar’s system. The functions \(\phi_n\) are defined by the equations

\[ \phi_1(t) = \phi_0(0) = 1 \quad \text{for} \ t \in [0,1] \]

and, for \(m = 2^n + k\) with \(1 \leq k \leq 2^n, \ n = 0, 1, \ldots, \)

\[ \phi_m(t) = \phi^K_n(t) = \begin{cases} +\sqrt{2^n} & \text{for} \ t \in \left[ \frac{2K - 2}{2^{n-1}}, \frac{2K - 1}{2^n + 1} \right], \\
-\sqrt{2^n} & \text{for} \ t \in \left[ \frac{2K - 1}{2^n + 1}, \frac{2K}{2^n + 1} \right], \\
0 & \text{elsewhere}. \end{cases} \]

P. L. Ulyanov [10] has obtained the following result.

**Theorem 1.** There exists a function \(f \in \cap_{1 < p < +\infty} L^p[0,1], \ f = \sum_{i=1}^{\infty} a_i \phi_i, \) and a permutation of the integers \(\pi\) such that the series \(\sum a_{\pi(i)} \phi_{\pi(i)}(t)\) diverges unboundedly on \([0,1]\) almost everywhere.

The first proof given in [10] uses a quite difficult modification of Zahorski’s construction. In a recent publication [11] P. L. Ulyanov gives a simpler proof (in \(L^2\)). From this article and the arguments of the first paper we can get a simpler proof of this theorem. Olevskii [15] gave also (in \(L^2\)) another proof of this result. From this theorem we can get the theorem announced.

**Theorem 2.** There exists a function \(f \in \cap_{1 < p < +\infty} L^p[0,1]\) and for each \(p, 1 < p < +\infty, \) an isomorphism \(T\) such that \(T\) and \(T^{-1}\) are both power bounded and

\[ M_n(T)f = \frac{I + T + T^2 + \cdots + T^{n-1}}{n} f \]

does not converge almost surely.
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Proof. Let \( p \) be fixed, \( 1 < p < +\infty \). From Haar's system \( (\phi^{(K)}_n) = (\phi_n) \) we can get an unconditional base of \( L^p[0,1] \), \( (\tilde{\phi}^{(K)}_n) = (\tilde{\phi}_n) \). Consider the permutation \( \pi \) and \( f \) of the previous theorem; \( (\tilde{\phi}^{(m)}_n) \) is again an unconditional base of \( L^p[0,1] \) and if \( f = \sum a_i \tilde{\phi}_i \) then \( f = \sum a_{\pi(i)} \tilde{\phi}_{\pi(i)} \).

By using Remark 1 in [7] we can get an isomorphism \( T (T \text{ and } T^{-1} \text{ are both power bounded}) \) and a sequence \( n_1 < n_2 < \cdots < n_K < \cdots \) such that \( \|M_{n_k}(T) - Q_K\| < +\infty \), where \( Q_K \) is the projection

\[
Q_K \left( \sum_{i=1}^{\infty} b_i \tilde{\phi}_i \right) = \sum_{i>K} b_i \tilde{\phi}_i.
\]

We can then deduce easily that the sequence \( M_{n_k}(T)f \) does not converge almost surely on \([0,1]\). This result extends easily to the case of operators defined by a regular sequence \((\alpha_n)\) (see Definition 1).

Theorem 3. Let \((\alpha_n)\) be a regular sequence and consider the sequence of operators

\[
R_n(T) = \sum_{K=0}^{\infty} \alpha_n K^T K.
\]

Then for each \( p, 1 < p < +\infty \), there exists an isomorphism \( T : L^p \to L^p \) and \( f \in L^p \) such that the sequence \( R_n(T)f \) does not converge almost surely.

Proof. Consider the unconditional base \( (\tilde{\phi}^{(m)}_n) \) of the previous proof and let \( P_m \) be the projection on \( \phi^{(m)}_n \). As in [12] the operator \( T \) will be diagonal, i.e., \( T = \sum \lambda_i \alpha_i P_i \), where the \( \lambda_i \) are complex numbers with \( |\lambda_i| = 1 \) and \( \lambda_i \neq 1 \), \( \forall i \).

Take \( \lambda_1, |\lambda_1| = 1 \) and \( \lambda_1 \neq 1 \). Choose \( (\varepsilon_i) \) such that \( \varepsilon_i > 0 \) and \( \sum \varepsilon_i = 1 \). There exist an integer \( n_1 \) and a real number \( \delta_1 \) such that \( |R_{n_1}(\lambda_1)| < \varepsilon_1 \) and \( |R_{n_1}(\lambda) - 1| < \varepsilon_1 \) if \( |\lambda - 1| < \delta_1 \), by the regular properties of the sequence \((\alpha_n)\).

If \( \lambda_i, n_i, \delta_i \) have been chosen for \( 1 < i < j \), then we choose \( \lambda_{j+1} \) such that \( |\lambda_{j+1} - 1| < \delta_j \) for \( 1 < i < j \). Then we take \( n_{j+1} > n_j \) such that \( |R_{n_{j+1}}(\lambda_i)| < \varepsilon_{j+1} \) for \( 1 < i < j + 1 \) and \( \delta_{j+1} > 0 \) such that \( |R_{n_{j+1}}(\lambda) - 1| < \varepsilon_{j+1} \) when \( |\lambda - 1| < \delta_{j+1} \).

The end of the proof is then the same as for the previous theorem.

Remark 4. (1) Theorem 2 shows that the Abel means of an isomorphism \( T \) such that \( T \text{ and } T^{-1} \) are power bounded in \( L^p \) are not always convergent a.e. (take \( \alpha_n = (1/n)(1 - 1/n)^K \)).

(2) An analog proof had been given in the case of contractions of \( L^2 \) in [13].

III. On the local ergodic theorem in \( L^p[0,1], 1 < p < +\infty \). Let \( (\tilde{\phi}^{(m)}_n) \) be the unconditional base of the previous section, \( V_K = I - Q_K \), and \( p \) fixed, \( 1 < p < +\infty \).

We have the following lemma, containing ideas from [9].

Lemma 5. There exist sequences \((\lambda_n), n = 1, \ldots, (t_K), K = 1, \ldots, \) with \( \lambda_n > 0 \) and \( 0 < t_K \to 0 \) such that \( \sum ||V_K - (1/t_K)S_K|| < +\infty \).

Proof. Choose \( (\varepsilon_k) \) \( \varepsilon_K > 0 \) such that \( \sum \varepsilon_k < +\infty \). Then take \( \lambda_1 > 0 \) arbitrarily and choose \( 0 < t_1 < 1 \) so small that

\[
\left| \frac{e^{i\lambda_1 t_1} - 1}{i\lambda_1 t_1} - 1 \right| < \varepsilon_1.
\]
If $\lambda_1, \lambda_2, \ldots, \lambda_K$ and $t_1, \ldots, t_K$ are already chosen, choose $\lambda_{K+1} > 0$ sufficiently large so that

$$\left| \frac{e^{i\lambda_{K+1}t_m} - 1}{i\lambda_{K+1}t_m} \right| < \varepsilon_m \quad \text{for each} \ m = 1, 2, \ldots, K.$$ 

Then choose $s < t_{K+1} < 1/(K+1)$ so small that

$$\left| \frac{e^{i\lambda_{K+1}s} - 1}{i\lambda_{K+1}s} - 1 \right| < \varepsilon_{K+1}.$$ 

**Theorem 6.** Given $p$, $1 < p < +\infty$, there exists a function $f \in \cap_{1 < p < +\infty} L^p[0,1]$ and a one-parameter semigroup $(T_t, t \geq 0)$ of power bounded operators in $L^p[0,1]$ for which $(1/t)f \int_0^t T_t f \, ds$ does not converge a.e. as $t \to 0^+$. 

**Proof.** Let $p$ fixed, $1 < p < +\infty$. Consider the one-parameter semigroup $T_t$ defined by

$$T_t \left( \sum_{n=1}^{\infty} \alpha_n \tilde{\phi}_n(n) \right) = \sum_{n=1}^{\infty} e^{i\lambda_n t} \alpha_n \tilde{\phi}_n(n)$$

where the $\lambda_n$ are those of the previous lemma and $\pi$ the permutation in Theorem 1. Then we have $T_0 = I$ (identity),

(i) for each $t, s$, $T_t \circ T_s = T_{t+s}$,
(ii) $\lim_{s \to t} \|T_tf - T_sf\|_p = 0 \forall t \geq 0$ and $\forall f \in L^p,$
(iii) $S_tf = \sum_{n=1}^{\infty} (1/i\lambda_n) (\exp(-i\lambda_n t) - 1) \alpha_n \tilde{\phi}_n(n) (t > 0).$

It is now easy to see that $(SK/tK)f$ does not converge almost everywhere for the function $f$ of Theorem 1.

**Remark 7.** It is known that for $T_0 = I$ and any semigroup of positive bounded linear operators on $L^p$ with $P(\Omega) < +\infty$ we can get the local ergodic theorem (see [8]).

**References**


14. ___, *Sur les opérateurs à puissances bornées et le théorème ergodique ponctuel dans $L^p[0,1]$, $1 < p < +\infty$* (a paraître).


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