ANALYTIC FUNCTIONALS AND THE BERGMAN PROJECTION ON CIRCULAR DOMAINS

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ABSTRACT. A property of the Bergman projection associated to a bounded circular domain containing the origin in $\mathbb{C}^N$ is proved: Functions which extend to be holomorphic in large neighborhoods of the origin are characterized as Bergman projections of smooth functions with small support near the origin. For certain circular domains $D$, it is also shown that functions which extend holomorphically to a neighborhood of $\overline{D}$ are precisely the Bergman projections of smooth functions whose supports are compact subsets of $D$. Two applications to proper holomorphic mappings are given.

This paper treats properties of the Bergman projection on certain domains in $\mathbb{C}^N$. We denote by $L^2(D)$ the space of functions which are square integrable with respect to Lebesgue measure on a domain $D$, and by $H^2(D)$ the (Bergman) space of holomorphic functions in $L^2(D)$. The orthogonal projection $P: L^2(D) \to H^2(D)$ is called the Bergman projection. The Bergman kernel is the integral kernel $k(z, w)$ such that, for all $f$ in $L^2(D)$,

$$Pf(z) = \int_D f(w)k(z, w) \, dw.$$ 

An analytic functional on a domain $D$ in $\mathbb{C}^N$ is a continuous linear functional on $\mathcal{O}(D)$, the space of holomorphic functions on $D$, with the topology of uniform convergence on compact subsets of $D$. The connection with the Bergman projection is

**Lemma 1.** Let $D$ be a domain in $\mathbb{C}^N$, $f$ in $H^2(D)$, and $U$ an open, relatively compact subset of $D$. There exists $u$ in $C^\infty_0(U)$ with $Pu = f$ if and only if, for some compact $K \subset U$ and some constant $C > 0$,

$$\left| \int_D \overline{f} g \right| \leq C\|g\|_K$$

holds for all $g$ in $H^2(D)$, where $\|g\|_K = \sup\{|g(z)|: z \in K\}$, and the integral is with respect to Lebesgue measure on $\mathbb{C}^N$.

**Proof.** Consider the linear functional $T_f: H^2(D) \to \mathbb{C}$ given by

$$T_f(g) = \langle g, f \rangle = \int_D \overline{f} g.$$ 

If (1) holds, then $T_f$ extends to a continuous linear functional on $C(K)$, the space of continuous functions on $K$. Hence $T_f$ is represented by a measure $\mu$ supported...
on $K$: 
\[ T_f(g) = \int_K g \, d\mu. \]

The measure $\mu$ is smoothed by convolution with a radially symmetric function of small support. The converse assertion follows easily from the orthogonality property of the Bergman projection. For details, see [6].

A circular domain in $\mathbb{C}^N$ is one invariant under the one-parameter family $z \rightarrow e^{it}z$ of biholomorphisms of $\mathbb{C}^N$. Let us recall several properties of such domains. First, if $D$ is bounded and contains the origin, then there exists an orthonormal basis $\{p_1, p_2, \ldots\}$ of $H_2(D)$ consisting of homogeneous polynomials. From the formula

\[ k(z, w) = \sum_{j=1}^{\infty} p_j(z)p_j(w), \]

it follows that for $z$ and $w$ in $D$ the identity $k(tz, w) = k(z, tw)$ holds for all complex $t$ for which it is meaningful. If $r < 1$ is such that $rD \subseteq D$, and $w$ in $rD$ is fixed, then $k(z, w) = k(z/r, rw)$ defines a holomorphic extension of $k(\cdot, w)$ to $(1/r)D$.

A related property of the Bergman kernel on a bounded circular domain, not necessarily containing the origin, is that for $w$ in a fixed compact subset of $D$, $k(\cdot, w)$ extends holomorphically to a domain containing $\overline{D}$. (See, e.g., [1].)

**Lemma 2.** Let $D$ be a bounded circular domain. There is a larger domain $D_1$, containing $D$ compactly, such that $O(D_1)$ is dense in $H_2(D)$.

**Remark.** When $D$ contains 0, the polynomials are dense in $H_2(D)$.

**Proof.** Let $U$ be any open set compactly contained in $D$. Let $S = \{Pu: u \in \mathcal{C}_0^\infty(U)\}$. Any $g$ in $S$ can be written $g(z) = \int_U u(w)k(z, w) \, dw$. By the extendibility property of $k(z, w)$ mentioned above, $g$ extends holomorphically to a larger domain $D_1$ which depends only on $U$. Thus it suffices to show that $S$ is dense in $H_2(D)$. Suppose $f$ in $H_2(D)$ is orthogonal to $S$. Then

\[ 0 = \int_D \overline{f}Pu = \int_D \overline{f}u \]

holds for all $u$ in $\mathcal{C}_0^\infty(U)$. Hence $f \equiv 0$ in $U$. This completes the proof.

The object of this paper is to prove the next two theorems:

**Theorem I.** Let $D \subseteq B (= \{|z| < 1\})$ be a circular domain such that, for some $a > 0$, $aB = \{|z| < a\} \subseteq D$. Let $r$ be real, $0 < r < a$. The following are equivalent:

1. $f|_D = Pu$ for some $u$ in $\mathcal{C}_0^\infty(rB)$,
2. $f$ in $O(r^{-1}B)$.

**Theorem II.** Let $D \subseteq B (= \{|z| < 1\})$ be a circular domain with boundary $bD$ of class $C^1$ such that, for every $p$ in $bD$, the complex line $\{tp: t \in \mathbb{C}\}$ intersects $bD$ transversally. The following are equivalent:

1. $f$ is in $O(\overline{D})$,
2. $f|_D = Pu$ for some $u$ in $\mathcal{C}_0^\infty(D)$.

**Remark.** Theorem I generalizes a result of Bell [1] for such domains: The restriction of a polynomial $f$ to $D$ can be written $f|_D = Pu$, where $u$ is a smooth polynomial.
function which can be chosen to have support contained in any neighborhood of the origin. That (2) implies (1) in Theorem II means that \( D \) satisfies condition \( Q \).

(Smooth bounded strictly pseudoconvex domains with real-analytic boundary also satisfy condition \( Q \). See [2].)

**Proof of Theorem I.** \((1) \Rightarrow (2)\). If \( u \) is in \( C^\infty_0(rB) \), then

\[
f(z) = Pu(z) = \int_D k(z, w)u(w)\,dw.
\]

Since \( k(\cdot, w) \) extends holomorphically to \( r^{-1}B \), so does \( f \).

\((2) \Rightarrow (1)\). Choose \( R < r \) so that \( f \) is in \( O(R^{-1}B) \), and define \( \psi(z) = \chi_{RD}R^{-2N}f(R^{-2}z) \). Then

\[
P\psi(z) = R^{-2N} \int_{RD} k(z, w)f(R^{-2}w)\,dw = \int_D k(z, Rt)f(R^{-1}t)\,dt
\]

\[
= \int k(Rz, t)f(R^{-1}t)\,dt = f(z),
\]

where the change of variable \( w = Rt \) was made. If \( g \) is in \( H_2(D) \), then by the orthogonality of the Bergman projection,

\[
\left\| \int_D \overline{f} g \right\| = \left\| \int_D \overline{P\psi} g \right\| = \left\| \int_{RD} \overline{g} \right\| \leq \text{volume}(RD)\|\psi\|_{RD}\|g\|_{RD} = C\|g\|_{RD}.
\]

This, together with Lemma 1, completes the proof of Theorem I.

**Remark.** The proof of Theorem II is similar to that of Theorem I except that more effort is required to verify the hypothesis of Lemma 1.

**Proof of Theorem II.** That \((2) \Rightarrow (1)\) follows immediately from the fact that \( k(z, w) \) extends (in \( z \)) across the boundary for \( w \) in a fixed compact subset of \( D \). The remainder of this paper is a proof that \((1) \Rightarrow (2)\).

**Lemma 3.** Let \( D \subset C^1 \) be a domain with real-analytic boundary. If \( f \) is in \( O(D) \), then there exist \( C > 0 \) and a compactum \( K \subset D \) so that, for all \( g \) in \( H_2(D) \),

\[
\left\| \int_D \overline{f} g \right\| \leq C\|g\|_K.
\]

**Proof.** See [6, Theorem III.8]. See also [2, Lemma 1], where a simple proof is given for domains in \( C^N \).

**Remark.** Lemma 3 is false for every \( C^2 \) but not real-analytic bounded domain \( D \) in \( C^1 \). (See [6, Theorem IV.3].) The situation in \( C^N \) is quite different: The domains in Theorem II can be \( C^2 \) yet far from smooth.

To continue the proof of the theorem, we wish to find \( C > 0 \) and \( K \subset D \) so that inequality (1) of Lemma 1 holds for all \( g \) in \( H_2(D) \). There exists a domain \( D_1 \supset \overline{D} \) so that \( O(D_1) \) is dense in \( H_2(D) \), so it suffices to show that (1) holds for all \( g \) in \( O(D_1) \). Since both sides of (1) are positively homogeneous in \( g \), we may assume further that \( \|g\|_D = 1 \) for some domain \( \overline{D} \) with \( D \subset \overline{D} \subset D_1 \). Let such \( g \) be called admissible; the set of admissible \( g \) is a normal family on \( D \).

We shall use Fubini's theorem for differential forms [5, p. 210] to estimate the integral in (1). Let \( \Pi: C^N \setminus \{0\} \to C^{P-1} \) be the projection given in homogeneous coordinates on \( C^{P-1} \) by \( \Pi(z_1, \ldots, z_N) = [z_1: \cdots: z_N] \). Let \( \omega \) be the fundamental
(1, 1)-form of the Fubini-Study metric on $\mathbb{C}P^{N-1}$. A computation (see [6, p. 382] for details) yields

$$(\Pi^*\omega)^{N-1} \wedge \partial \overline{\partial}(|z|^2) = (i/2\pi)^{N-1}(N-1)!|z|^{2-2N} \, dz_1 \wedge \overline{dz_1} \cdots dz_N \wedge \overline{dz_N}.$$ 

It follows that if $H$ is any measurable function defined on $D$,

$$(N-1)! \left( \frac{2}{i\pi^{N-1}} \right) \int_D H \, dm_{2N}$$

$$= (N-1)! \left( \frac{i}{2\pi} \right)^{N-1} \int_{D\setminus\{0\}} H \, dz_1 \wedge \overline{dz_1} \cdots dz_N \wedge \overline{dz_N}$$

$$= \int_{\alpha \in \mathbb{C}P^{N-1}} \left\{ \int_{\Pi^{-1}(\alpha) \cap D} H |z|^{2N-2} \partial \overline{\partial}(|z|^2) \right\} \omega^{N-1},$$

where $dm_{2N}$ is Lebesgue measure on $\mathbb{C}^N$. For fixed $\alpha$ in $\mathbb{C}P^{N-1}$, fix $u$ in $\Pi^{-1}(\alpha)$ with $|u| = 1$. Then $\Pi^{-1}(\alpha) = \{\tau u : \tau \in \mathbb{C}\setminus\{0\}\}$. Taking $\tau$ to be the coordinate on $\Pi^{-1}(\alpha)$ gives

$$\int_{\Pi^{-1}(\alpha) \cap D} H |z|^{2N-2} \partial \overline{\partial}(|z|^2) = \frac{2}{i} \int_{\alpha \cap D} H(\tau) |\tau|^{2N-2} \, dm_2(\tau),$$

where $dm_2(\tau)$ is Lebesgue measure on the complex line $\alpha$.

We apply this formula in the case $H = \overline{f}\,g$, where $f$ and $g$ are as above. Let $\alpha_0$ in $\mathbb{C}P^{N-1}$ be fixed. Since $D$ is circular, $D_{\alpha_0} \equiv D \cap \Pi^{-1}(\alpha_0)$ is also circular—in particular, $D_{\alpha_0}$ has real-analytic boundary. Since the restrictions of $f$ and $g$ to $D_{\alpha_0}$ are holomorphic, we may conclude from Lemma 3 that

$$\left| \int_{D_{\alpha_0}} \overline{f} g |\tau|^{2N-2} \, dm_2(\tau) \right| = \left| \int_{D_{\alpha_0}} \overline{f} \tau^{N-1} g \tau^{N-1} \, dm_2(\tau) \right|$$

$$\leq C_{\alpha_0} \|g\|_{D(\alpha_0,\varepsilon)} \|f\|_{D(\alpha_0,1/\varepsilon)} < N_{\alpha_0} \|g\|_{D(\alpha_0,1/N_{\alpha_0})}$$

holds for all admissible $g$ and some integer $N_{\alpha_0}$, where $D(\alpha_0,\varepsilon) = \{z \in D_{\alpha_0} : \text{dist}(z,bD_{\alpha_0}) \geq \varepsilon\}$. The second inequality holds because $D$ is compact.

The inequality

$$\left(2\right) \quad \left| \int_{D_\alpha} \overline{f} g r^{2N-2} \, dm_2 \right| < N_{\alpha_0} \|g\|_{D(\alpha_1/\alpha_0)}$$

is valid for $\alpha = \alpha_0$ and arbitrary admissible $g$. For a fixed $g$, both sides of (2) are continuous in $\alpha$, so there is a neighborhood $U_{\alpha_0}$ of $\alpha_0$ in $\mathbb{C}P^{N-1}$ such that (2) holds for all $\alpha$ in $U_{\alpha_0}$.

Since the admissible $g$ are uniformly equicontinuous on $\overline{D}$, the neighborhoods $U_{\alpha_0}$ can be chosen so that (2) holds for all admissible $g$ and all $\alpha$ in $U_{\alpha_0}$. Since $\mathbb{C}P^{N-1}$ is compact, finitely many such neighborhoods $U_{\alpha_1}, \ldots, U_{\alpha_k}$ form an open cover. Taking $M = \max\{N_{\alpha_1}, \ldots, N_{\alpha_k}\}$, we have

$$\left(3\right) \quad \left| \int_{D_\alpha} \overline{f} g r^{2N-2} \, dm_2 \right| \leq M \|g\|_{D(\alpha_1/M)}$$

for all $\alpha$ and all admissible $g$. 

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Since all the $D_{\alpha}$ meet $bD$ transversally, $K = \bigcup_{\alpha \in CP^{N-1}} D(\alpha, 1/M)$ is relatively compact in $D$. Using (3),
\[
\left| \int_D \bar{f} g \right| = \frac{\pi^{N-1}}{(N-1)!} \left| \int_{CP^{N-1}} \left\{ \int_{D_{\alpha}} \bar{f} g r^{2N-2} dm \right\} \omega^{N-1} \right| \\
\leq \frac{\pi^{N-1}}{(N-1)!} \left| \int_{CP^{N-1}} M \|g\|_K \omega^{N-1} \right| \\
\leq \frac{\pi^{N-1}}{(N-1)!} M \|g\|_K \left| \int_{CP^{N-1}} \omega^{N-1} \right| .
\]

By the Wirtinger theorem, $\left(1/(N-1)!\right) \int_{CP^{N-1}} \omega^{N-1}$ is the (finite) volume of $CP^{N-1}$. This completes the proof of Theorem II.

We conclude with two applications to proper holomorphic mappings.

**Corollary 1.** Suppose that $D_1$ is a smooth bounded domain satisfying condition $Q$, and that $D_2$ is as in Theorem II. If $f$ is a proper holomorphic mapping of $D_1$ onto $D_2$, then $f$ extends to be holomorphic in a neighborhood of $D_1$.

**Proof.** The proof is identical to that of Theorem 1 in [2], except that our Theorem II replaces Bell's Lemma 1.

It is interesting to compare Corollary 1 to Theorem 2 in [3]: A proper holomorphic mapping of one complete Reinhardt domain onto another extends to be holomorphic in a neighborhood of the first.

**Corollary 2.** Let $D_1$ be a bounded strictly pseudoconvex domain with real-analytic boundary, and $D_2$ a bounded smooth simply connected circular domain which satisfies the transversality condition of Theorem II. If $f$ is a proper holomorphic mapping of $D_1$ onto $D_2$, then $f$ extends to a biholomorphism between larger domains containing $D_1$ and $D_2$.

**Remark.** Corollary 2 generalizes a result of Bell [4]: Let $D_1$ be as in Corollary 2, $D_2$ a smooth bounded pseudoconvex complete Reinhardt domain. If $f$ is a proper holomorphic mapping of $D_1$ onto $D_2$, then $f$ extends to be a biholomorphism of larger domains.

**Proof.** The proof is the same as in [4], except that Fact 2 follows from Theorem II.

**References**


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