COMPOSING FUNCTIONS OF BOUNDED \( \varphi \)-VARIATION

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ABSTRACT. Let \( F_n \) be finite-valued functions on \((-\infty, \infty)\), \( F_n(0) = 0 \), \( n = 1, 2, \ldots \). For \( x \in \mathcal{V}_\varphi(a,b) \), the class of functions of bounded \( \varphi \)-variation, the compositions \( F_n(x) \) are studied. The main result of this paper is Theorem 1 stating necessary and sufficient conditions for the sequence \( \text{var}_\varphi(F_n(x),a,b) \) to be bounded for each \( x \in \mathcal{V}_\varphi(a,b) \) (\( \varphi \) denotes here another \( \varphi \)-function).

1. Throughout this paper, by \( \varphi \)-function we understand a continuous, unbounded, nondecreasing function on \((0, \infty)\), with \( \varphi(u) = 0 \) iff \( u = 0 \). Such a function is said to satisfy condition \( \Delta_2 \) (for small \( u \)) whenever \( \varphi(2u) \leq k\varphi(u) \) with some constant \( k > 0 \) for \( 0 \leq u \leq u_0 \). We denote by \( X \) the vector space of real-valued functions on \((a, b)\) such that \( x(a) = 0 \).

For a given partition \( \pi: a = t_0 < t_1 < \cdots < t_n = b \), let us form the variational sum

\[
\sigma_\varphi(x, \pi) = \sum_{i=1}^{n} \varphi(|x(t_i) - x(t_{i-1})|), \quad x \in X.
\]

The number

\[
\text{var}_\varphi(x, a, b) = \text{var}_\varphi(x) = \sup_{\pi} \sigma_\varphi(x, \pi),
\]

where the supremum is taken over all \( \pi \), is called the \( \varphi \)-variation of \( x \) on \((a, b)\).

The following classes of functions will be considered: \( \mathcal{V}_\varphi = \{x \in X: \text{var}_\varphi(x) < \infty\} \) and \( \mathcal{V}_\varphi^* = \{x \in X: \text{var}_\varphi(\lambda x) < \infty \text{ for some } \lambda > 0\} \). We will also write \( \mathcal{V}_\varphi(a, b) \) and \( \mathcal{V}_\varphi^*(a, b) \). \( \mathcal{V}_\varphi^* \) is a vector space. Whenever each element \( x \in \mathcal{V}_\varphi^* \) satisfies the so-called condition B.1 [5, p. 50] i.e.

\[
\text{var}_\varphi(\lambda x) \to 0 \quad \text{as } \lambda \to 0,
\]

a generated norm can be defined in \( \mathcal{V}_\varphi^* \),

\[
||x||_{\mathcal{V}_\varphi^*} = \inf\{\varepsilon > 0: \text{var}_\varphi(x/\varepsilon) \leq \varepsilon\},
\]

and then \( \mathcal{V}_\varphi^* \) is complete in this \( F \)-norm. When \( \varphi \) is a \( \varphi \)-function of the form \( \varphi(u) = \psi(u^s) \), \( 0 < s \leq 1 \), where \( \psi \) is a convex \( \varphi \)-function, then condition B.1 is satisfied for each \( x \in \mathcal{V}_\varphi^* \). In this case, one can define in \( \mathcal{V}_\varphi^* \), along with the generated norm, an \( s \)-homogeneous norm

\[
||x||_{s,\mathcal{V}_\varphi^*} = \inf\{\varepsilon > 0: \text{var}_\varphi(x/\varepsilon^{1/s}) \leq 1\};
\]

\( \| \cdot \|_{s,\mathcal{V}_\varphi} \) and \( \| \cdot \|_{\mathcal{V}_\varphi^*} \) are equivalent.

Recall that the notion of functions of bounded \( \varphi \)-variation appeared first in the papers of Wiener [6] (for \( \varphi(u) = u^2 \)), J. Marcinkiewicz [3] and L. C. Young [7] (for...
\( \varphi(u) = u^p, \ p \geq 1 \). The last author introduced this idea for an unrestricted \( \varphi \)-function \([8]\). The spaces of functions of bounded \( \varphi \)-variation were studied from the point of view of fundamental notions of functional analysis and some applications by J. Musielak and W. Orlicz \([4]\) and R. \.Lesniewicz and W. Orlicz \([2]\).

2. THEOREM 1. Let \( \varphi \) be an arbitrary \( \varphi \)-function, \( \psi \) a \( \varphi \)-function satisfying \( \Delta_2 \) for small \( u \). Let \( F_n \) be functions on \( (-\infty, \infty) \), \( F_n(0) = 0 \), \( n = 1, 2, \ldots \), such that

(a) \( \sup_n \text{var}_\psi(F_n(x), a, b) \leq \infty \) for \( x \in \mathcal{V}_\varphi(a, b) \).

Then for every \( v > 0 \) there exists a constant \( K_v \) such that the inequality

(b) \( \psi(|F_n(u_2) - F_n(u_1)|) \leq K_v \varphi(|u_2 - u_1|) \) holds for \( u_2, u_1 \in (-v, v) \), \( n = 1, 2, \ldots \).

Conversely, (b) implies (a).

PROOF. To show (a) \( \Rightarrow \) (b), let us first observe that (a) implies the uniform boundedness in common of \( F_n(u) \) in \( (-v, v) \). If the sequence \( (F_n(u)) \) were not uniformly bounded in common in \( (-v, v) \), then for some sequence of indices \( n \) there would exist \( u_i \in (-v, v) \) and \( u_0 \), such that

(1) \( F_n(u_i) \to \infty \),

and

(2) \( \varphi(2|u_i - u_0|) \leq 1/2^i, \quad i = 1, 2, \ldots \).

We can additionally assume \( u_0 \leq u_i \). Choose arbitrarily points \( a < t_0 < t_1 < t_2 < \cdots < t_n < \cdots < b \) and define the function \( x \) by taking \( x(a) = 0, x(t_0) = u_0, x(t_i) = u_i, x(t) = u_0 \) in the remaining points of \( (a, b) \). Since for \( j > i \), \( \varphi(|u_i - u_j|) \leq 1/2^{i-1} \), we can easily calculate by (2) that \( \text{var}_\varphi(x) < \infty \). Since \( |F_n(x(t_i))| = |F_n(u_i)| \), then in view of (1), we have a contradiction with (a).

If condition (b) is not satisfied, then there exist indices \( n \) (not necessarily all different, they may even be almost all equal) and intervals \( (u_i, v_i) \subset (-v, v) \) such that

(3) \( k_i = \frac{\psi(F_n(u_i) - F_n(x_i))}{\varphi(v_i - u_i)} \to \infty \).

Since \( F_{n_i} \) are uniformly bounded in common in \( (-v, v) \),

(4) \( v_i - u_i \to 0 \).

Choose \( w_i = (u_i + v_i)/2 \). Passing, if necessary, to a partial sequence one can assume \( w_i \to w_0 \). Let us consider both possible cases.

1°. \( w_0 \) is contained in an at most finite number of intervals \( (u_i, v_i) \). Assume, for instance, that infinitely many intervals not containing \( w_0 \) lie to the right of \( w_0 \). The reasoning would be analogous if infinitely many of them were to the left of \( w_0 \). Let us define by induction a partial sequence of these intervals \( (u_i', v_i') \) having the following properties:

(\( \alpha \)) \( w_0 < u_{i+1} < v_{i+1} < u_i' < v_i' \), \( i = 1, 2, \ldots \),

(\( \beta \)) \( \varphi(w_0 - v_i') \leq 1/2^i \), \( i = 1, 2, \ldots \),

(\( \gamma \)) \( k_i' > 2^{2i+1} \), \( i = 1, 2, \ldots \).

Here \( k_i' \) means the \( k_i \) corresponding to the interval \( (u_i', v_i') \) in (3), and \( n_i \) is replaced
by \( n'_i \). From (\(\beta\)) it follows that \( \varphi(v'_i - u'_i) \leq 1/2^i \). We determine integers \( m_i \) so that
\[
\frac{1}{2^{i+1}} < m_i \varphi(v'_i - u'_i) \leq \frac{2}{2^i}, \quad i = 1, 2, \ldots,
\]
and groups of points in \((a, b)\) in such a way that
\[
\tilde{t}^{i+1}_1 < \tilde{t}^{i+1}_2 < \tilde{t}^{i+1}_3 < \cdots < \tilde{t}^{i+1}_{m_{i+1}} < \tilde{t}^{i+1}_1
\]
and
\[
< \tilde{t}^i_1 < \tilde{t}^i_2 < \tilde{t}^i_3 < \cdots < \tilde{t}^i_{m_i} < \tilde{t}^i_1,
\]
\[i = 1, 2, \ldots, t^{i}_{m_i} \to a, \tilde{t}^i_{m_i} > a.
\]
Define the function
\[
x(t) = \begin{cases} 
0 & \text{for } t = a, \\
v'_i & \text{for } t'_i, t'_2, \ldots, t'_m, \quad i = 1, 2, \ldots, \\
\tilde{v}'_i & \text{for } \tilde{t}^i_1, \tilde{t}^i_2, \ldots, \tilde{t}^i_{m_i}, \quad i = 1, 2, \ldots, \\
v'_j & \text{for } t'_j < t < t'_j, \quad j = 1, 2, \ldots, m_i - 1, \\
\tilde{v}'_j & \text{for } \tilde{t}^i_j < t < \tilde{t}^i_j, \quad j = 1, 2, \ldots, m_i - 1, \\
w_0 & \text{for } t^{i}_{m_i} < t < t^{i+1}_{m_i}, \quad t^i_{m_i} < t \leq b, \quad i = 1, 2, \ldots.
\end{cases}
\]
Both (\(\beta\)) and (5) imply \( \text{var}_\varphi(x) < \infty \). However, taking into consideration (5) and (\(\gamma\)), we have
\[
\psi(|F_{n'_i}(x(t'_1)) - F_{n'_i}(x(t'_2))|) + \psi(|F_{n'_i}(x(t'_2)) - F_{n'_i}(x(t'_3))|) + \cdots
\]
\[+ \psi(|F_{n'_i}(x(\tilde{t}^i_{m_i})) - F_{n'_i}(x(t^i_{m_i}))|)
\]
\[= m_i \psi(|F_{n'_i}(v'_i) - F_{n'_i}(u'_i)|) = k'_i m_i \varphi(v'_i - u'_i)
\]
\[\geq 2^{2i+1} \frac{1}{2^{i+1}} = 2^i \to \infty \quad \text{as } i \to \infty.
\]
Hence \( \text{var}_\varphi(F_{n'_i}(x)) \to \infty \) and we have a contradiction.

2°. \( w_0 \) is contained in infinitely many \((u_i, v_i)\). Let \( u_i < w_0 < v_i \) for some \( i \).
Then
\[
S_i = \sup \left( \frac{\psi(|F_{n_i}(v_i) - F_{n_i}(w_0)|)}{\varphi(v_i - w_0)}, \frac{\psi(|F_{n_i}(w_0) - F_{n_i}(u_i)|)}{\varphi(w_0 - u_i)} \right)
\]
\[\geq \frac{1}{2k} \psi(|F_{n_i}(v_i) - F_{n_i}(u_i)|),
\]
where \( k \) is a constant such that \( \psi(2u) \leq k\psi(u) \), when \( 0 \leq u \leq 2c \), with \( |F_{n_i}(u)| \leq c \) for \(-v \leq u \leq v\), \( i = 1, 2, \ldots \). This is so because the following inequalities hold:
\[
\psi(|F_{n_i}(v_i) - F_{n_i}(u_i)|) \leq \psi(2|F_{n_i}(v_i) - F_{n_i}(w_0)|) + \psi(2|F_{n_i}(w_0) - F_{n_i}(u_i)|)
\]
\[\leq kS_i(\varphi(v_i - w_0) + \varphi(w_0 - u_i)) \leq 2kS_i \varphi(v_i - u_i).
\]
So, making use of (3) and (6) we can exhibit an infinite sequence of intervals of the form \((u'_i, w_0)\) or \((w_0, v'_i)\) and appropriate \( n'_i \) such that either
\[
\frac{\psi(|F_{n'_i}(v'_i) - F_{n'_i}(w_0)|)}{\varphi(v'_i - w_0)} \geq 2^{2i+1}
\]
or
\[
\frac{\psi(|F_{n'_i}(w_0) - F_{n'_i}(u'_i)|)}{\varphi(w_0 - u'_i)} \geq 2^{2i+1}, \quad i = 1, 2, \ldots.
\]
Assume, for instance, (7) holds. The proof of (7') would be analogous. Passing, if necessary, to a partial sequence we can assume $\varphi(v_i' - w_0) \leq 1/2^i, \ i = 1, 2, \ldots$. Define integers $m_i$ so that

$$
\frac{1}{2^{i+1}} \leq m_i \varphi(v_i' - w_0) \leq \frac{2}{2^i}, \ i = 1, 2, \ldots.
$$

Define groups of points as in (T) and the function

$$
x(t) = \begin{cases} 0 & \text{for } t = a, \\ v_i' & \text{for } t = t_1^i, t_2^i, \ldots, t_{m_i}^i, \ i = 1, 2, \ldots, \\ w_0 & \text{elsewhere in } (a, b). \end{cases}
$$

It follows from (8) that $\text{var}_{\varphi}(x) < \infty$. However, in view of (7) we have

$$\varphi(|F_n(x(t_1^i) - x(t_1^i))|) + \varphi(|F_n(x(t_2^i) - x(t_2^i))|) + \cdots + \varphi(|F_n(x(t_{m_i}^i) - x(t_{m_i}^i))|)
= m_i \varphi(|F_n(v_i') - F_n(w_0)|) \geq 2^{i+1} \frac{1}{2^{i+1}} = 2^i \to \infty.
$$

Hence $\sup_{n_i} \text{var}_{\varphi}(F_n(x)) = \infty$ and we have a contradiction. Similarly, we get a contradiction if $w_0 = u_i$ or $w_0 = v_i$ for infinitely many $i$.

To prove (b)$\Rightarrow$(a), note that if $x \in \mathcal{V}_{\varphi}(a, b)$, then for some $v$, $x(t) \in (-v, v)$, $t \in (a, b)$ and for an arbitrary partition $\pi: a = t_0 < t_1 < \cdots < t_k = b$ we have

$$
\sum_{i=1}^{k} \varphi(|F_n(x(t_i)) - F_n(x(t_{i-1}))|) \leq K_\varphi \sum_{i=1}^{k} \varphi(|x(t_i) - x(t_{i-1})|),
$$

so

$$
\text{var}_{\varphi}(F_n(x)) \leq K_\varphi \text{var}_{\varphi}(x), \quad n = 1, 2, \ldots.
$$

Thus (a) holds. □

A sequence of elements $(x_n) \in \mathcal{V}_{\varphi}(a, b)$ is called $n$-convergent (two-norm convergent) to $x_0$ if $x_n(t) \to x_0(t)$ uniformly in $(a, b)$, $\text{var}_{\varphi}(x_n) \leq k, n = 1, 2, \ldots$. $n$-convergence in $\mathcal{V}_{\varphi}(a, b)$ is defined analogously.

**COROLLARY.** (b) implies the following property of equicontinuity of $F_n(x)$ with respect to $n$-convergence:

If the sequence $x_n \in \mathcal{V}_{\varphi}(a, b)$ is $n$-convergent in $\mathcal{V}_{\varphi}(a, b)$ to $x = 0$, then $F_n(x_n)$ is $n$-convergent to 0 in $\mathcal{V}_{\varphi}(a, b)$.

Indeed, if $x_n(a) = 0$, $\text{var}_{\varphi}(x_n) \leq k, n = 1, 2, \ldots$, and $x_n(t) \to 0$ uniformly in $(a, b)$, then the functions $x_n$ are uniformly bounded in common in some $(-v, v)$ and by (b) we get $\varphi(|F_n(x_n(t))|) \leq K_\varphi \varphi(|x_n(t)|)$, so $F_n(x_n(t)) \to 0$ uniformly in $(a, b)$. The second part of our assertion is, in view of (9), obvious. □

Note that it follows immediately from (b) and (9) that if the sequence $(x_n)$ is $n$-convergent to $x_0$ in $\mathcal{V}_{\varphi}(a, b)$, then the sequence $(F(x_n))$ is $n$-convergent to $F(x_0)$ in $\mathcal{V}_{\varphi}(a, b)$. From Theorem 1 it follows also that if for every $x \in \mathcal{V}_{\varphi}(a, b)$, $F(x) \in \mathcal{V}_{\varphi}(a, b)$, then this operator is continuous with respect to modular convergence: $\text{var}_{\varphi}(x_n - x_0) \to 0$ implies $\text{var}_{\varphi}(F(x_n) - F(x_0)) \to 0$. 

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3. THEOREM 2. Let \( \phi \) be a \( \varphi \)-function satisfying the condition \( \Delta_2 \) for small \( u \) and \( F \) a function on \( (-\infty, \infty) \), \( F(0) = 0 \).

A. The operator \( F(x) \in \mathcal{V}_\phi(a,b) \) for each \( x \in \mathcal{V}_\phi(a,b) \) if and only if for every \( v > 0 \), there exists a constant \( K_v \) such that

\[
\phi(|F(u_2) - F(u_1)|) \leq K_v \phi(|u_2 - u_1|) \quad \text{for} \quad u_1, u_2 \in (-v, v).
\]

B. Under the additional assumption that \( \phi \) is strictly increasing and \( \varphi^{-1} \) satisfies \( \Delta_2 \) for small \( u \), the inequality (*) is equivalent to

\[
|F(u_2) - F(u_1)| \leq K_v |u_2 - u_1| \quad \text{for} \quad u_1, u_2 \in (-v, v).
\]

On the above assumptions it follows from (**) that for arbitrary \( \varphi \)-function \( \psi \), \( x \in \mathcal{V}_\psi^*(a,b) \Rightarrow F(x) \in \mathcal{V}_\psi^*(a,b) \).

PROOF. Ad. A. Inequality (*) is an immediate consequence of Theorem 1, if we put \( F_n = F, \; n = 1, 2, \ldots \).

Ad. B. Under the additional assumptions on \( \phi \) we get from (**)

\[
|F(u_2) - F(u_1)| \leq \varphi^{-1}(K_v \phi(|u_2 - u_1|)) \leq K_v |u_2 - u_1|.
\]

Let \( x \in \mathcal{V}_\psi^* \), i.e., \( \text{var}_\psi(\lambda x) \leq \infty \) for some \( \lambda > 0 \). For an arbitrary partition \( \pi: a = t_0 < t_1 < \cdots < t_n = b \) and some \( v \) we get from (**)

\[
\sum_{i=1}^n \psi \left( \frac{\lambda}{K_v} |F(x(t_i)) - F(x(t_{i-1}))| \right) \leq \sum_{i=1}^n \psi(\lambda |x(t_i) - x(t_{i-1})|) \leq \text{var}_\psi(\lambda x).
\]

Hence \( \text{var}_\psi((\lambda/K_v)F(x)) \leq \infty \); that is \( F(x) \in \mathcal{V}_\psi^* \).

COROLLARY. If for any \( x \in \mathcal{V}(a,b) \) there is \( F(x) \in \mathcal{V}(a,b) \), then for arbitrary \( \varphi \)-function, \( x \in \mathcal{V}_\phi^*(a,b) \) implies \( F(x) \in \mathcal{V}_\psi^*(a,b) \).

This Corollary and Theorem 2 generalize, in a sense, a result of M. Josephy [1].

REMARK. A. If \( \psi \) is of the form \( \psi(u) = \chi(u^s), \; \chi \) convex, \( 0 < s \leq 1 \), and \( \psi \) satisfies \( \Delta_2 \) for small \( u \), then the assumption (a) of Theorem 1, \( \sup_n \text{var}_\psi(F_n(x)) \leq \infty \) for \( x \in \mathcal{V}_\phi(a,b) \), is equivalent to \( \sup_n \|F_n(x)\|_{s,\psi} < \infty \). This last assumption is reminiscent of the one concerning sequences of linear operators in normed spaces from the Banach-Steinhaus Uniform Boundedness Principle. From this assumption follows (b) of Theorem 1, so we have (9) and thus, if \( \varphi(x) \leq 1 \), then with some constant \( C \) the inequality

\[
\text{var}_\psi(F_n(x)) \leq C \text{var}_\phi(x) \leq C
\]

holds for \( n = 1, 2, \ldots \). Hence, taking \( C \geq 1 \),

\[
\text{var}_\psi \left( \frac{F_n(x)}{C^{1/s}} \right) \leq 1,
\]

that is to say \( \|F_n(x)\|_{s,\psi} \leq C \) when \( \varphi(x) \leq 1 \), \( n = 1, 2, \ldots \). Thus we obtained a conclusion analogous to the assertion of the Banach-Steinhaus theorem.

B. Put \( F_n(u) = u, \; n = 1, 2, \ldots \). In this case the condition (b) is, with some \( K > 0 \), \( u_0 > 0 \), equivalent to \( \psi(u) \leq K \varphi(u) \) for \( 0 \leq u \leq u_0 \). This is the known necessary and sufficient condition for the inclusion \( \mathcal{V}_\phi(a,b) \subset \mathcal{V}_\psi(a,b) \) to hold. (However, in Theorem 1 it has been additionally assumed that \( \psi \) satisfies \( \Delta_2 \) for small \( u \).)
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