

## A NECESSARY CONDITION FOR $L^2$ STABILITY OF QUASILINEAR CONSERVATION LAWS

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ABSTRACT. This paper proves: It is the necessary condition for  $L^2$  stability of quasilinear conservation laws that the solution is absolutely continuous.

Consider quasilinear conservation laws

$$(1) \quad u_t + f(u)_x = 0, \quad -\infty < x < +\infty,$$

where  $u(x, t) \in \mathbf{R}^n$  and  $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is a smooth nonlinear function.

Function  $u(x, t) \in L^\infty(\Omega) \cap BV(\Omega)$ ,  $\Omega \in \mathbf{R}^2$  is called a weak solution if  $u$  satisfies (1) in the sense of distributions. Here  $BV(\Omega)$  denotes the space of functions whose first-order derivatives are locally finite Borel measures [7].

It is well known that in general the initial value problem to (1) does not have a globally defined smooth solution; on the other hand, the uniqueness is lost in the broader class of above weak solutions. In order to single out an admissible solution, some criterion is needed. A number of such criteria have been proposed, e.g., Lax's [4], Liu's [5], viscosity [6], entropy [3] and entropy rate [1] criteria, which are all equivalent when applied to weak shocks with genuinely nonlinear fields.

In [2], R. J. DiPerna discussed the uniqueness of solutions to (1) under the entropy criterion.

A function  $\eta: \mathcal{D} \rightarrow \mathbf{R}$  defined on an open domain  $\mathcal{D} \subset \mathbf{R}^n$  is called an entropy for (1) with entropy flux  $q: \mathcal{D} \rightarrow \mathbf{R}$ , if an additional conservation law

$$(2) \quad \eta(u)_t + q(u)_x = 0$$

holds for all smooth solutions of (1). According to the entropy criterion a weak solution  $u(x, t)$  with range in  $\mathcal{D}$  is admissible if it satisfies

$$(3) \quad \eta(u)_t + q(u)_x \leq 0$$

in the sense of distributions. Under the entropy criterion, the establishing of uniqueness and stability of solutions to (1) generally depends on some  $L^2$  inequalities [2].

As to the  $L^2$  stability of (1), R. J. DiPerna proved the following theorem [2]:

**THEOREM 1 (DIPERNA).** *Suppose that  $w(x, t)$  is a Lipschitz continuous solution and  $u(x, t)$  is an admissible weak solution of (1) in  $\mathcal{S}(T) = \{(x, t): 0 \leq t < T\}$ . Then*

$$(4) \quad \int_{|x| \leq M} |u(x, t) - w(x, t)|^2 dx \leq C_2 \int_{|x| \leq M + C_1 t} |u(x, 0) - w(x, 0)|^2 dx,$$

where the constant  $C_1$  depends on  $f$  and  $L^\infty$ -norms of  $w$  and  $u$  while the constant  $C_2$

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depends on  $f$ ,  $T$ , the  $L^\infty$ -norms of  $w$  and  $u$  and the Lipschitz constant of  $w(x, 0)$ .

The above theorem says that the Lipschitz continuous solution of (1) is  $L^2$  stable in the class of admissible weak solutions.

In this paper, we prove conversely that if the solution of (1) is  $L^2$  stable, then it must be absolutely continuous. Therefore, it is impossible to establish  $L^2$  stability for any discontinuous solutions of (1).

Our main result is

**THEOREM 2.** *Suppose the weak solution  $u(x, t)$  of (1) is  $L^2$  stable and  $u_0(x) \equiv u(x, 0)$  is locally Lipschitz continuous. Then  $u$  is absolutely continuous.*

**PROOF.** Clearly,  $u(x + \varepsilon, t)$  is also a solution of (1) for  $\varepsilon > 0$ . By (4) we have

$$\int_{|x| \leq M} |u(x + \varepsilon, t) - u(x, t)|^2 dx \leq C_2 \int_{|x| \leq M + C_1 t} |u_0(x + \varepsilon) - u_0(x)|^2 dx$$

and hence

$$\begin{aligned} \int_{|x| \leq M} \frac{|u(x + \varepsilon, t) - u(x, t)|^2}{\varepsilon^2} dx &\leq C_2 \int_{|x| \leq M + C_1 t} \frac{|u_0(x + \varepsilon) - u_0(x)|^2}{\varepsilon^2} dx \\ &\leq C_2 (M + C_1 T) (\text{Lip } u_0)^2 = \text{const} \end{aligned}$$

where  $\text{Lip } u_0$  is the Lipschitz constant of  $u_0(x)$  for  $|x| \leq M + C_1 T$ . In view of the weak compactness of Banach space  $L^2$ , letting  $\varepsilon \rightarrow 0^+$ , we get

$$\int_{|x| \leq M} |u_x|^2 dx \leq \text{const.}$$

By using the Cauchy inequality we obtain

$$(5) \quad \int_{|x| \leq M} |u_x| dx \leq \text{const.}$$

The estimate (5) implies that  $u(x, t)$  is an absolutely continuous solution. This completes the proof.

We note that in our theorem the system (1) may be neither genuinely nonlinear nor strictly hyperbolic.

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