A NECESSARY CONDITION FOR $L^2$ STABILITY OF QUASILINEAR CONSERVATION LAWS

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Abstract. This paper proves: It is the necessary condition for $L^2$ stability of quasilinear conservation laws that the solution is absolutely continuous.

Consider quasilinear conservation laws

$$u_t + f(u)_x = 0, \quad -\infty < x < +\infty,$$

where $u(x, t) \in \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^n$ is a smooth nonlinear function.

Function $u(x, t) \in L^\infty(\Omega) \cap BV(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ is called a weak solution if $u$ satisfies (1) in the sense of distributions. Here $BV(\Omega)$ denotes the space of functions whose first-order derivatives are locally finite Borel measures [7].

It is well known that in general the initial value problem to (1) does not have a globally defined smooth solution; on the other hand, the uniqueness is lost in the broader class of above weak solutions. In order to single out an admissible solution, some criterion is needed. A number of such criteria have been proposed, e.g., Lax's [4], Liu's [5], viscosity [6], entropy [3] and entropy rate [1] criteria, which are all equivalent when applied to weak shocks with genuinely nonlinear fields.

In [2], R. J. DiPerna discussed the uniqueness of solutions to (1) under the entropy criterion.

A function $\eta: \mathcal{D} \to \mathbb{R}$ defined on an open domain $\mathcal{D} \subset \mathbb{R}^n$ is called an entropy for (1) with entropy flux $q: \mathcal{D} \to \mathbb{R}$, if an additional conservation law

$$\eta(u)_t + q(u)_x = 0$$

holds for all smooth solutions of (1). According to the entropy criterion a weak solution $u(x, t)$ with range in $\mathcal{D}$ is admissible if it satisfies

$$\eta(u)_t + q(u)_x \leq 0$$

in the sense of distributions. Under the entropy criterion, the establishing of uniqueness and stability of solutions to (1) generally depends on some $L^2$ inequalities [2].

As to the $L^2$ stability of (1), R. J. DiPerna proved the following theorem [2]:

**Theorem 1 (DiPerna).** Suppose that $w(x, t)$ is a Lipschitz continuous solution and $u(x, t)$ is an admissible weak solution of (1) in $\mathcal{D}(T) = \{(x, t): 0 \leq t < T\}$. Then

$$\int_{|x| \leq M} |u(x, t) - w(x, t)|^2 dx \leq C_2 \int_{|x| \leq M} |u(x, 0) - w(x, 0)|^2 dx,$$

where the constant $C_1$ depends on $f$ and $L^\infty$-norms of $w$ and $u$ while the constant $C_2$...
depends on $f$, $T$, the $L^\infty$-norms of $w$ and $u$ and the Lipschitz constant of $w(x, 0)$.

The above theorem says that the Lipschitz continuous solution of (1) is $L^2$ stable in the class of admissible weak solutions.

In this paper, we prove conversely that if the solution of (1) is $L^2$ stable, then it must be absolutely continuous. Therefore, it is impossible to establish $L^2$ stability for any discontinuous solutions of (1).

Our main result is

**Theorem 2.** Suppose the weak solution $u(x, t)$ of (1) is $L^2$ stable and $u_0(x) \equiv u(x, 0)$ is locally Lipschitz continuous. Then $u$ is absolutely continuous.

**Proof.** Clearly, $u(x + \epsilon, t)$ is also a solution of (1) for $\epsilon > 0$. By (4) we have

$$\int_{|x| < M} |u(x + \epsilon, t) - u(x, t)|^2 \, dx \leq C_2 \int_{|x| < M + C_1 T} |u_0(x + \epsilon) - u_0(x)|^2 \, dx$$

and hence

$$\int_{|x| < M} \frac{|u(x + \epsilon, t) - u(x, t)|^2}{\epsilon^2} \, dx \leq C_2 \int_{|x| < M + C_1 T} \frac{|u_0(x + \epsilon) - u_0(x)|^2}{\epsilon^2} \, dx$$

$$\leq C_2 (M + C_1 T) (\text{Lip } u_0)^2 = \text{const}$$

where Lip $u_0$ is the Lipschitz constant of $u_0(x)$ for $|x| \leq M + C_1 T$. In view of the weak compactness of Banach space $L^2$, letting $\epsilon \to 0^+$, we get

$$\int_{|x| < M} |u_x|^2 \, dx \leq \text{const}.$$

By using the Cauchy inequality we obtain

$$\int_{|x| < M} |u_x| \, dx \leq \text{const}.$$  

The estimate (5) implies that $u(x, t)$ is an absolutely continuous solution. This completes the proof.

We note that in our theorem the system (1) may be neither genuinely nonlinear nor strictly hyperbolic.

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**References**


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