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EXTENSION OF A THEOREM OF BAAYEN AND HELMBERG ON MONOTHETIC GROUPS

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ABSTRACT. Let G and K be compact monothetic groups and let ϕ be a continuous homomorphism from G onto K. If k is a generator of K, must there exist a generator g of G such that $\phi(g) = k$? A useful theorem of Baayen and Helmberg provides an affirmative answer if K is the circle T. We show that the answer remains affirmative as long as K is metrizable. We also provide an example to show that the answer may be negative for nonmetrizable K.

A Hausdorff topological group G is said to be *monothetic* if and only if G contains an element which generates a dense subgroup of G. Such an element is called a *generator* of G. A monothetic group is automatically abelian, and a classical result of Weil asserts that a locally compact monothetic group must either be topologically isomorphic to the group of integers with the discrete topology or else must be compact (see §§9.1 and 9.2 of [**5**]). The multiplicative group T of all complex numbers of modulus 1, when given its usual topology, is a compact monothetic group; indeed, it is easily seen that if θ is a real number, $\exp(i\theta)$ is a generator of T if and only if θ is irrational. Now let G be a compact monothetic group and let γ be a continuous homomorphism from G onto T. It is clear that if x is a generator of G, then $\gamma(x)$ is a generator of T. The question naturally arises: If $\exp(i\theta)$ is a generator of T, must there exist a generator x of G such that $\gamma(x) = \exp(i\theta)$? This question was answered in the affirmative in an important and useful result of Baayen and Helmberg (see Lemma 1 of [**3**]; simplified proofs may be found in [**1**], Theorem 5.11 of [**2**], and Lemma 4.1 of [**6**]). It is the purpose of this note to investigate how far this result may be generalized. We shall prove the following theorem, already announced in §5.30 of [**2**]:

THEOREM. *Let G and K be compact monothetic groups, and let K be metrizable. Let ϕ be a continuous homomorphism from G onto K. Then if k is a generator of K there exists a generator g of G such that $\phi(g) = k$.*

We shall also give an example to show that the conclusion may fail if K is not metrizable.

In our proof (which, incidentally, does not assume the Baayen-Helmberg result) we shall employ some of the simpler aspects of Pontryagin's duality theory for Hausdorff locally compact abelian (LCA) groups as presented, say, in [**5**]. In our case we shall be dealing, however, only with compact and discrete groups. Fundamental is the fact that a compact LCA group G is monothetic if and only if its (discrete) dual $\hat{G}$ is isomorphic to a subgroup of T_d, by which we mean T with the discrete topology (see §24.32 of [**5**]). Also important is the fact that an element x of an

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LCA group G is a generator of G if and only if $\gamma(x) \neq 1$ for all $\gamma \neq 1$ in $\hat{G}$; i.e., if and only if x, when considered as a character on $\hat{G}$, is one-one (see §25.11 of [**5**]). We shall also make use of the adjoint ϕ^* of a continuous homomorphism ϕ between LCA groups (see §§24.37–24.41 of [**5**]). We begin our proof by establishing three purely algebraic results. The reader may consult [**4**] and Appendix A of [**5**] as general references for abelian group theory. If A and B are subgroups of an abelian group C, then we write $C = A \oplus B$ to indicate that C is the direct sum of A and B.

LEMMA 1. *Let A and B be finite or countably infinite isomorphic subgroups of T_d. Let C and D be subgroups of T_d such that $A \oplus C = B \oplus D = T_d$. Then C and D are isomorphic.*

PROOF. Since T_d is divisible, the same is true of A, B, C and D. If we apply the structure theorem for divisible groups (see A.14 of [**5**]) to all five groups under consideration, a simple cardinality argument (we omit the details) reveals that C and D have the same p-ranks and torsion-free rank and are therefore isomorphic. (Note that the result may fail if A is uncountable.)

LEMMA 2. *Let A be a finite or countably infinite subgroup of T_d. If ψ is a monomorphism from A into T_d, then ψ can be extended to an isomorphism $\psi^{\#}$ of T_d.*

PROOF. Since T_d is divisible, we may find a minimal divisible subgroup $\overline{A}$ of T_d containing A (see Theorem 24.4 of [**4**]). Since $\overline{A}$ has the same rank as A (p. 107 of [**4**]), $\overline{A}$ is either trivial or countably infinite. By Theorem 21.1 of [**4**] we may extend $\psi: A \to T_d$ to a homomorphism $\bar{\psi}: \overline{A} \to T_d$. It follows from Lemmas 24.2 and 24.3 of [**4**] that $\bar{\psi}$ is also a monomorphism. Since both $\overline{A}$ and $\bar{\psi}(\overline{A})$ are divisible there exist subgroups C and D of T_d such that $\overline{A} \oplus C = \bar{\psi}(\overline{A}) \oplus D = T_d$ (see Theorem 21.2 of [**4**]). Since $\overline{A}$ is at most countably infinite and $\overline{A} \cong \bar{\psi}(\overline{A})$, it follows from Lemma 1 that $C \cong D$. Let f be any isomorphism from C onto D. Then define $\psi^{\#}: T_d \to T_d$ by this rule: If $t \in T_d$, write $t = \bar{a} + c$ uniquely for $\bar{a} \in \overline{A}$ and $c \in C$ and set $\psi^{\#}(t) = \bar{\psi}(\bar{a}) + f(c)$. It is easy to verify that $\psi^{\#}$ is an isomorphism from T_d onto T_d which extends ψ.

LEMMA 3. *Let B be a finite or countably infinite abelian group. Suppose that f_1 and f_2 are monomorphisms from B into T_d. Then there exists an isomorphism g of T_d such that $f_2 = g \circ f_1$.*

PROOF. Set $A = f_1(B)$ and define a mapping $\psi: A \to T_d$ by this rule: For $a \in A$, $\psi(a) = f_2(b)$ where b is the unique element of B such that $a = f_1(b)$. It is immediately verified that ψ is a monomorphism. Now let g be the isomorphism $\psi^{\#}$ guaranteed to exist by Lemma 2. Then for each $b \in B$ we have $g(f_1(b)) = \psi(f_1(b)) = f_2(b)$; i.e., $g \circ f_1 = f_2$, as desired.

Now we proceed with the proof of the Theorem. As Kuipers and Niederreiter point out in their simplification of the proof of Baayen and Helmberg's result, we may assume that $G = (T_d)\hat{}$. (This follows immediately from the fact that any compact monothetic group is a quotient of $(T_d)\hat{}$ by a closed subgroup, since $\hat{G}$ is a subgroup of T_d.) Suppose, then, that ϕ is a continuous homomorphism from $(T_d)\hat{}$ onto K, and let k be a generator of K. We seek a generator g of $(T_d)\hat{}$ (that

is, a one-one character g of T_d) such that $\phi(g) = k$. Let $\phi^*: \hat{K} \to T_d$ be the adjoint of ϕ. (Here we have identified $((T_d)^\frown)^\frown$ with T_d.) By §24.41 of [**5**] ϕ^* is one-one. Moreover, the generator k of K may be regarded as a monomorphism from $\hat{K}$ into T_d. The metrizability condition on K entails that $\hat{K}$ is finite or countably infinite (§24.15 of [**5**]). Now set $B = \hat{K}$, $f_1 = \phi^*$ and $f_2 = k$ in Lemma 3. We conclude that there exists an isomorphism $g: T_d \to T_d$ such that $k = g \circ \phi^*$. Since g is one-one, it is a generator of $(T_d)^\frown$, and we need only verify that $\phi(g) = k$. This will be done if we show that $\phi(g)$ and k agree as characters on $\hat{K}$. To this end, let $\gamma \in \hat{K}$. We have

$$(\phi(g))(\gamma) = \gamma(\phi(g)) = (\phi^*(\gamma))(g) = g(\phi^*(\gamma)) = (g \circ \phi^*)(\gamma) = k(\gamma),$$

so that $\phi(g) = k$. (A quicker, if perhaps less perspicuous, way to see this is by using §24.41(a) of [**5**]: We have $\phi(g) = (\phi^*)^*(g) = g \circ \phi^* = k$.) This completes the proof of the Theorem.

We now show that the conclusion of the Theorem may fail if K is not metrizable. Let D be a *proper* subgroup of T_d such that $D \cong T_d$. (That such a D exists may be seen by recalling that T_d is a direct sum of quasicyclic groups, one for each prime p, and continuum many copies of the group Q of rational numbers; just remove one of the Q's to obtain D.) Let $i: D \to T_d$ be the injection and let f be any isomorphism from D *onto* T_d. Then $\hat{D}$ is compact and monothetic and i^* is a continuous homomorphism from $(T_d)^\frown$ *onto* $\hat{D}$. Now f may be regarded as a generator of $\hat{D}$. We claim, however, that there is no generator g of $(T_d)^\frown$ such that $i^*(g) = f$. For assume that such a g exists. Then g is a one-one character on T_d. To say that $i^*(g) = f$ is to say that $f = g \circ i$. But since i is not surjective, there exists $t \in T_d$ such that $t \notin i(D)$. Now $g(t) \in T_d$ and $f: D \to T_d$ is surjective, so there exists $d \in D$ such that $f(d) = g(t)$. But then $g(t) = f(d) = (g \circ i)(d) = g(i(d))$. Since g is one-one, we have $t = i(d)$, contradicting the fact that $t \notin i(D)$.

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