AN OPEN BOOK DECOMPOSITION FOR $RP^2 \times S^1$

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Abstract. In this paper an open book decomposition for $RP^2 \times S^1$ is exhibited.

In a recent paper [1], Berstein and Edmonds prove the existence of open book decompositions for compact, nonorientable 3-manifolds. Their method is to pull back an open book decomposition of $RP^2 \times S^1$ via a branched cover. Their demonstration that $RP^2 \times S^1$ has such a decomposition is algebraic; they define a homomorphism $\pi_1(RP^2 \times S^1, \text{ knot}) \to \mathbb{Z}$ and apply Stallings' fibering theorem [2]. In this paper I explicitly exhibit an open book decomposition of $RP^2 \times S^1$ via a map $RP^2 \times S^1, \text{ same knot} \to S^1$, inducing the same homomorphism on $\pi_1$.

Let $D^2 = \{ z \in \mathbb{C} : |z| \leq 1 \}$. Think of $RP^2$ as $D^2$ with antipodal points of the boundary circle identified. Define $\varphi : D^2 \to D^2$ by $\varphi(z) = -z$. Note that $\varphi$ induces a map $\varphi : RP^2 \to RP^2$ which is isotopic to the identity (just rotate through $180^\circ$). Thus we may take $D^2 \times I$ with the identifications

$$D^2 \times I \to RP^2 \times I \to RP^2 \times S^1/(x, 0) \sim (\varphi(x), 1)$$

as our model of $RP^2 \times S^1$. The image of $\{ -\frac{1}{2} \} \times I \cup \{ \frac{1}{2} \} \times I$ is a circle $C$ in $RP^2 \times S^1$.

Let $p : (D^2 - \{ -\frac{1}{2}, \frac{1}{2} \}) \times I \to S^1$ by

$$p(z, t) = \frac{z^2 - 1/4}{|z^2 - 1/4|} e^{2\pi it}.$$

An open book decomposition of $RP^2 \times S^1$ is given by taking as binding the circle $C$ and as pages the surfaces $F_\theta$ in $RP^2 \times S^1$ represented by $\{ -\frac{1}{2}, \frac{1}{2} \} \times I \cup p^{-1}(e^{2\pi i \theta})$ in $D^2 \times I$.

In Figure 1 I exhibit the surface $F = F_1$ in $RP^2 \times S^1$ by drawing its intersection with each $D^2 \times \{ t \} \in D^2 \times I$. The boundary of $F$ is $C$.

$F \cap D^2 \times \{ \frac{1}{2} \}$ is not a manifold, because $F$ has a saddle at this level (i.e., a critical point with respect to the Morse function $\text{proj} : RP^2 \times S^1 \to S^1$). Because the only critical point is of index one, $\chi(F) = -1$. Thus $F$, being nonorientable, is a punctured Klein bottle. Because $F$ is the graph of a function from $D^2 - \{ -\frac{1}{2}, \frac{1}{2} \} \to [0, 1)$, we see that $RP^2 \times S^1 - C$ is foliated by leaves which are simply copies of

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$F$ translated in the $S^1$ direction. Indeed these are the surfaces $F_{\theta}$, where $F = F_1$. This gives an open book decomposition of $RP^2 \times S^1$ with binding $C$. The intersection of this foliation with an arbitrary $D^2 \times \{t\}$ is shown in Figure 2.

**Figure 2**

**References**


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