PROPERTIES OF ENOMORPHISM RINGS
OF MODULES AND THEIR DUALS

SOUMAYA MAKDISSI KHURI

Abstract. Let $RM$ be a nonsingular left $R$-module whose Morita context is
nondegenerate, let $B = \text{End}_RM$ and let $M^* = \text{Hom}_R(M, R)$. We show that $B$ is
left (right) strongly modular if and only if any element of $B$ which has zero kernel in
$RM (M^*_R)$ has essential image in $RM (M^*_R)$, and that $B$ is a left (right) Utumi ring if
and only if every submodule $RU$ of $RM (U^*_R$ of $M^*_R)$ such that $U = 0 (U^* = 0)$ is
essential in $RM (M^*_R)$.

1. Introduction. Let $RM$ be a left $R$-module whose standard Morita context is
nondegenerate (see Definition 1); let $B = \text{End}_RM$ be the ring of $R$-endomorphisms
of $RM$ and let $M^* = \text{Hom}_R(M, R)$ be its dual module. Then $B$ is left nonsingular if
and only if $RM$ is nonsingular (i.e. $M$ satisfies the following: any $m \in M$ with
essential annihilator in $R$ must be zero), and $B$ is right nonsingular if and only if
$M^*_R$ satisfies the following condition: If $U^*_R$ is an essential submodule of $M^*_R$ then
the annihilator of $U^*$ in $B$ must be zero (Proposition 5). This condition certainly
holds if $M^*_R$ is nonsingular. Of course, just as for $RM$, $M^*_R$ is nonsingular if and only
if $\text{End}_RM^*$ is right nonsingular. Our concern, however, is with $B$, which is in general
—for example for a nonfinitely generated $RM$—a proper subring of $\text{End}_RM^*$;
hence a condition on $RM$ which is equivalent to a certain left property of $B$ is not
expected to be equivalent to the same right property of $B$ when it is reflected in $M^*_R$.
In this paper, we investigate this situation and try to pick out some left-right
properties of $B$ which are symmetrically, or almost symmetrically, represented on
$RM$ and $M^*_R$. For example, we find that $B$ is left strongly modular if and only if any
element of $B$ which has zero kernel in $RM$ has essential image in $RM$, while $B$ is
right strongly modular if and only if any element of $B$ which has zero kernel in $M^*_R$
has essential image in $M^*_R$ (Theorem 3); and we find that $B$ is a left Utumi ring if
and only if every submodule $RU$ of $RM$ such that $U = 0$ is essential in $RM$, while $B$
is a right Utumi ring if and only if every submodule $U^*_R$ of $M^*_R$ such that $U^* = 0$ is
essential in $M^*_R$ (Theorem 7). These conditions naturally raise the general question
of how $B$ sits in $\text{End}_RM^*$, a question which we do not treat in this paper, but which
we expect to investigate in a future article.
2. Preliminaries. The left and right annihilators in $B$ of a subset $K$ of $B$ will be denoted by $\mathcal{L}(K)$ and $\mathcal{R}(K)$, respectively. The notation $l_M(K)$, $r_M(K)$, $r_B(U)$, $l_B(U^*)$ will be used for the annihilators in $M$ of $K \subseteq B$ in $M^*$ of $K \subseteq B$, in $B$ of $U \subseteq M$ and in $B$ of $U^* \subseteq M^*$, respectively. Also, for $RU \subseteq R M$ and $U^* \subseteq M^*_R$, we will use: $I_B(U) = \{ b \in B: Mb \subseteq U \}$ and $I_B(U^*) = \{ b \in B: bM^* \subseteq U^* \}$. The notation $\mathcal{R} U \subseteq \mathcal{C}_R M$ will be used to indicate that $U$ is an essential $R$-submodule of $M$, i.e. $U$ intersects nontrivially every nonzero $R$-submodule of $M$. Recall that $\mathcal{R} M$ is said to be nonsingular in case, for $m \in M$, $l_R(m) \subseteq \mathcal{R} R \Rightarrow m = 0$; $B$ is said to be left (right) nonsingular if $\mathcal{R} B$ is nonsingular.

We recall the following definition and proposition from [4]:

**Definition 1.** Let $(R, M, N, S)$ be a Morita context; that is, let $R M_S$ and $S N_R$ be bimodules with an $R-R$ bimodule homomorphism $(\cdot, \cdot): M \otimes_S N \rightarrow R$ and an $S-S$ bimodule homomorphism $[\cdot, \cdot]: N \otimes_R M \rightarrow S$ satisfying

$$m x [n x, m_2] = (m x, n x)m_2 \quad \text{and} \quad n x (m_1, m_2) = [n_1, m_1]n_2$$

for all $m_1, m_2 \in M$ and $n_1, n_2 \in N$.

Then $(R, M, N, S)$ is said to be nondegenerate if and only if the four modules $R M$, $M_S$, $S N$, $N_R$ and the two pairings are faithful (the latter meaning that $(m, N) = 0$ implies $m = 0$, and three analogous implications).

This is equivalent to the eight natural maps associated with the Morita context being injective (for example, two of these maps are: $m \mapsto (m, -)$ and $r \mapsto (n \mapsto nr) \in \text{End}(S N)$, for $m \in M$, $n \in N$ and $r \in R$). The standard context $(R, M, M^*, B)$ of a module $R M$ is nondegenerate if and only if $R M$ is torsionless and faithful and the right annihilator of trace $(\mathcal{R} M)$ is zero. We shall call such a module—i.e. one whose standard context is nondegenerate—a nondegenerate module, for brevity.

**Proposition 1 [4, Proposition 14].** If the context $(R, M, N, S)$ is nondegenerate, and if one of $R R$, $R M$, $S N$, $S S$ is nonsingular, then all of them are nonsingular.

Henceforth, unless otherwise indicated, let $R M$ be a nondegenerate, nonsingular left $R$-module. Then, by the preceding, $R M$, $M_B$, $B M^*$, $M^*_R$ and the two pairings are faithful, and $R R$, $R M^*$ and $B B$ are nonsingular. $(\cdot, \cdot)$ and $[\cdot, \cdot]$ will denote the pairings associated with the standard context for $R M$, i.e. $(\cdot, \cdot)$ is defined by $(m, f) = mf$ for $m \in M$ and $f \in M^*$, and $[f, m]$ is defined by $m_i[f, m] = (m_i, f)m$ for all $m, m_i \in M$ and $f \in M^*$.

If $RU$ is a submodule of $R M$ then $[M^*, U]$ indicates the left ideal of $B$: $[M^*, U] = \{ \sum_{i=1}^n [m_i^*, u_i]: m_i^* \in M^*, u_i \in U \}$, and similarly for $[U^*, M]$ where $U^*_R$ is a submodule of $M^*_R$. Also, $U^* = \{ m^* \in M^*: (U, m^*) = 0 \}$ and $U^* = \{ m \in M: (m, U^*) = 0 \}$.

The well-known fact that, for a nonsingular module $R M$, any $R$-homomorphism $f$, to $R M$ from any other $R$-module, which has essential kernel is zero, will be used repeatedly without comment.

The following lemma will be useful to us in the sequel.

**Lemma 2.** For $K \subseteq B$, $\mathcal{L}(K) = l_B I_M(K) = l_B (KM^*)$, and $\mathcal{R}(K) = r_B (MK) = I_B r_{M^*} (K)$. 
Proof.

\(b \in \mathcal{L}(K) \iff bK = 0 \iff MbK = 0 \iff Mb \subseteq l_M(K) \iff b \in l_B l_M(K);\)

\(b \in \mathcal{R}(K) \iff Kb = 0 \iff bKM* = 0 \iff b \in l_B(KM*);\)

\(b \in \mathcal{S}(K) \iff Kb = 0 \iff MKb = 0 \iff b \in r_B(MK);\)

\(b \in \mathcal{S}(K) \iff Kb = 0 \iff KbM* = 0 \iff bM* \subseteq r_M*(K) \iff b \in l_B r_M*(K). \quad \square\)

3. Strongly modular and Utumi endomorphism rings. In [2], a Baer *-ring \(B\) is called strongly modular in case, for all \(b \in B\), \(\mathcal{S}(b) = 0\) implies that \(bB\) is essential in \(B\). Because of the involution, the definition is left-right symmetric. In the absence of an involution, call a ring \(B\) left strongly modular if, for \(b \in B\), \(\mathcal{L}(b) = 0 \Rightarrow bB \subseteq \epsilon B\), and right strongly modular if \(\mathcal{R}(b) = 0 \Rightarrow bB \subseteq \epsilon B\). It turns out that the properties of left and right strong modularity of \(B = \text{End}_R M\) are equivalent to almost symmetric conditions on \(R M\) and \(M^*_R\).

Theorem 3. (i) \(B\) is left strongly modular if and only if, for each \(b \in B\),

\(l_M(b) = 0 \Rightarrow Mb \subseteq \epsilon R M;\)

(ii) \(B\) is right strongly modular if and only if, for each \(b \in B\), \(r_{M^*}(b) = 0 \Rightarrow bM^* \subseteq \epsilon M^*_R.\)

Proof. By comparing the definition of left strong modularity with the condition on \(R M\) in (i), it is easily seen that (i) will follow as soon as we show that “\(\mathcal{L}(b) = 0\)” is equivalent to “\(l_M(b) = 0\)” and that “\(bB \subseteq \epsilon B\)” is equivalent to “\(MB \subseteq \epsilon R M\)”;

these equivalences will be proved in Lemma 4 which follows. Similarly, (ii) will follow once we show, in Lemma 4, that “\(\mathcal{R}(b) = 0\)” is equivalent to “\(r_{M^*}(b) = 0\)” and that “\(bB \subseteq \epsilon B\)” is equivalent to “\(bM^* \subseteq \epsilon M^*_R\)”.

Lemma 4. (i) For any subset \(K\) of \(B\), \(\mathcal{L}(K) = 0\) if and only if \(l_M(K) = 0\) and \(\mathcal{R}(K) = 0\) if and only if \(r_{M^*}(K) = 0\).

(ii) For any left ideal \(_B H\) of \(B\), \(_B H \subseteq \epsilon B\) if and only if \(MH \subseteq \epsilon R M\); and for any right ideal \(J_B\) of \(B\), \(J_B \subseteq \epsilon B\) if and only if \(JM^* \subseteq \epsilon M^*_R.\)

Proof. (i) Let \(K \subseteq B\) and consider the submodule \(l_M(K)\) of \(R M\). If \(l_M(K) \neq 0\), let \(0 \neq m \in l_M(K)\); then, by nondegeneracy, there is \(m* \in M^*\) such that \([m*, m] \neq 0\). Then, since \(M_B\) is faithful, \(0 \neq M[m*, m] = (M, m*)m \subseteq Rm \subseteq l_M(K)\); that is, \(0 \neq [m*, m] \in l_B l_M(K)\). Hence, \(l_M(K) = 0\) if and only if \(\mathcal{L}(K) = l_B l_M(K) = 0\).

Similarly, if \(0 \neq m* \in r_{M^*}(K)\), then nondegeneracy gives \(m \in M\) such that \([m*, m] \neq 0\), and since \(BM^*\) is faithful, \([m*, m]\) is a nonzero element of \(l_B r_{M^*}(K)\); hence \(r_{M^*}(K) = 0\) if and only if \(\mathcal{R}(K) = l_B r_{M^*}(K) = 0\).

(ii) Let \(_B H\) be an essential left ideal of \(B\) and let \(0 \neq m \in M\). Then \([M^*, m] \cap H \neq 0\), and, since \(M_B\) is faithful,

\(0 \neq M([M^*, m] \cap H) \subseteq M[M^*, m] \cap MH = (M, M^*)m \cap MH \subseteq Rm \cap MH,\)

proving that \(MH \subseteq \epsilon R M\).

Conversely, assume that \(MH \subseteq \epsilon R M\) for some left ideal \(_B H\) of \(B\) and let \(0 \neq c \in B\). Then, since \(M_B\) is faithful, \(Mc \neq 0\), and hence \(Mc \cap MH \neq 0\). By nondegeneracy,

\(0 \neq [M^*, Mc \cap MH] \subseteq [M^*, Mc] \cap [M^*, MH] \subseteq Bc \cap [M^*, MH].\)

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
This shows that \([M^*, MH] \subseteq cB_B\), and hence, since \([M^*, MH] \subseteq B_B\), we have \(H \subseteq cB_B\).

Similarly, if \(J\) is an essential right ideal of \(B\) and \(0 \neq m^* \in M^*\), then, by nondegeneracy, \([m^*, M] \neq 0\), hence \([m^*, M] \cap J \neq 0\). Since \(M^*_B\) is faithful, this implies

\[
0 \neq ([m^*, M] \cap J)M^* \subseteq [m^*, M]M^* \cap JM^*
\]

\[
= m^*(M, M^*) \cap JM^* \subseteq mR_B \cap JM^*;
\]

so \(JM^* \subseteq cM^*_R\).

Conversely, if \(JM^* \subseteq cM^*_R\) for some right ideal \(J_B\) of \(B\), and \(0 \neq c \in B\), then \(JM^* \cap cM^* \neq 0\) and \([JM^* \cap cM^*, M] \neq 0\) by nondegeneracy; hence,

\[
0 \neq [JM^* \cap cM^*, M] \subseteq [JM^*, M] \cap [cM^*, M] \subseteq [JM^*, M] \cap cB.
\]

This implies that \([JM^*, M] \subseteq cB_B\), and hence, since \([JM^*, M] \subseteq J\), we have \(J_B \subseteq cB_B\).

**Remarks.**

1. One property of nondegenerate modules that can be deduced from the proof of Lemma 4 is that \(I_R(U) = 0\) if and only if \(U = 0\) for a submodule \(RU\) of \(R_M\), and similarly for \(UR\) in \(RM\).

2. In the proof of Lemma 4(ii), we have shown that \(RU \subseteq cR_M \Rightarrow [M^*, U] \subseteq cB_B\) and \(UR \in cM^*_R \Rightarrow [U*, M] \subseteq cB_B\).

Aside from completing the proof of Theorem 3, Lemma 4 is also useful in giving a condition on \(M^*_R\) which is equivalent to right nonsingularity of \(B\), as in the next result.

**Proposition 5.** \(B\) is right nonsingular if and only if, for any submodule \(U^*_R\) of \(M^*_R\), \(U^*_R \subseteq cM^*_R \Rightarrow I_R(U^*) = 0\).

**Proof.** It was shown in [3, Proposition 1] that, under our present hypotheses, \(B\) is right nonsingular if and only if, for any submodule \(RU\) of \(R_M\), \(r_B(U) \subseteq cB_B \Rightarrow U = 0\).

Assume that \(B\) is right nonsingular and suppose that \(U^*_R \subseteq cM^*_R\); then, as noted in Remark 2 above, \([U^*, M] \subseteq cB_B\). We have \((Ml_B(U^*))[U^*, M] = (Ml_B(U^*), U^*)M = 0\); therefore \([U^*, M] \subseteq r_B(Ml_B(U^*))\), which implies \(r_B(Ml_B(U^*)) \subseteq cB_B\). Hence, by [3, Proposition 1], since \(B\) is right nonsingular, this implies that \(Ml_B(U^*) = 0\); hence, since \(M^*_B\) is faithful, we have \(I_R(U^*) = 0\).

Conversely, assume that \(U^*_R \subseteq cM^*_R\) implies \(I_R(U^*) = 0\). Suppose that \(RU\) is a submodule of \(R_M\) such that \(r_B(U) \subseteq cB_B\). Then, by Lemma 4(ii), \(U^*_R = r_B(U)M^* \subseteq cM^*_R\). Hence, by hypothesis, \(I_B(r_B(U)M^*) = 0\). But \(I_B(U) \subseteq I_B(r_B(U)M^*)\) since, always, \(I_B(U)r_B(U) = 0\); hence \(I_B(U) = 0\), which, by nondegeneracy (see Remark 1), implies that \(U = 0\), completing the proof.

A ring \(B\) is said to be a left Utumi ring in case, for any left ideal \(B_H\) of \(B\), \(\mathcal{O}(B_H) = 0 \Rightarrow B_H \subseteq cB_B\); \(B\) is called a right Utumi ring if, for any right ideal \(J_B\) of \(B\), \(\mathcal{L}(J_B) = 0 \Rightarrow J_B \subseteq cB_B\). In [2], it is shown that a strongly modular Baer *-ring is left and right Utumi [2, Theorem 2.3]. In our situation, i.e. for \(B = \text{End}_R M\), where \(RM\) is nondegenerate and nonsingular, it is easily shown that a left and right strongly modular Baer ring satisfies the Utumi conditions for principal left and right
ideals. In fact, $B$ need not be a Baer ring to show this; rather, left and right nonsingularity of $B$ is sufficient, with left and right strong modularity, in order to obtain the Utumi conditions for principal ideals, as can be seen in Proposition 6. For the full Utumi conditions on $B$, however, we can give, in Theorem 7, conditions on $\mathcal{R}$ and $\mathcal{M}_R$ which appear quite symmetrical.

**Proposition 6.** If $\mathcal{R}$ is such that $B = \text{End}_\mathcal{R}M$ is a left and right nonsingular, left and right strongly modular ring, then

(i) $\mathcal{L}(bB) = 0 \Rightarrow bB \subseteq \mathring{e}_B B$, and (ii) $\mathcal{R}(Bb) = 0 \Rightarrow Bb \subseteq \mathring{e}_B B$.

**Proof.** (i)

$\mathcal{L}(bB) = \mathcal{R}(b) = 0 \Rightarrow bB \subseteq \mathring{e}_B B$ since $B$ is left strongly modular,

$\Rightarrow \mathcal{R}(Bb) = 0$ since $B$ is left nonsingular,

$\Rightarrow bB \subseteq \mathring{e}_B B$ since $B$ is right strongly modular.

(ii)

$\mathcal{R}(Bb) = \mathcal{R}(b) = 0 \Rightarrow bB \subseteq \mathring{e}_B B$ since $B$ is right strongly modular,

$\Rightarrow \mathcal{L}(bB) = 0$ since $B$ is right nonsingular,

$\Rightarrow Bb \subseteq \mathring{e}_B B$ since $B$ is left strongly modular. □

For our last result, $\mathcal{R}$ is assumed to satisfy the standing hypothesis, i.e. $\mathcal{R}$ is nonsingular and nondegenerate.

**Theorem 7.** (i) $B = \text{End}_\mathcal{R}M$ is a left Utumi ring if and only if, for any submodule $\mathcal{R}U$ of $\mathcal{R}M$, $U \perp = 0 \Rightarrow \mathcal{R}U \subseteq \mathring{e}_\mathcal{R}M$; and (ii) $B = \text{End}_\mathcal{R}M$ is a right Utumi ring if and only if, for any submodule $U^*_\mathcal{R}$ of $\mathcal{M}_R$, $U^* = 0 \Rightarrow U^*_\mathcal{R} \subseteq \mathring{e}_\mathcal{M}_R$.

**Proof.** (i) Assume that $B$ is a left Utumi ring; then, by [3, Lemma 3], we have, for any submodule $\mathcal{R}X$ of $\mathcal{R}M$, $r_B(\mathcal{R}X) = 0 \Rightarrow \mathcal{R}X \subseteq \mathring{e}_\mathcal{R}M$. Let $\mathcal{R}U$ be a submodule of $\mathcal{R}M$ such that $U \perp = 0$. Then $b \in r_B(\mathcal{R}U) \Rightarrow Ub = 0 \Rightarrow (U, bm^*) = (Ub, m^*) = 0$, for each $m^* \in M^*$, $= bm^* = 0$ since $U \perp = 0$; but this means $bM^* = 0$, therefore, since $B$ is faithful, $b = 0$. Hence $r_B(U) = 0$, which implies, since $B$ is left Utumi, that $rU \subseteq \mathring{e}_\mathcal{R}M$.

Conversely, assume that $rU \perp = 0 \Rightarrow rU \subseteq \mathring{e}_\mathcal{R}M$ for every $rU \subseteq \mathcal{R}M$. Let $rH$ be a left ideal of $B$ with $\mathcal{R}(rH) = 0$. If $(MH, m^*) = 0$, then $(M, Hm^*) = 0$, which implies $HM^* = 0$ by nondegeneracy. Then $[Hm^*, M] = 0$, i.e. $H[m^*, M] = 0$, which implies $[m^*, M] = 0$. Again by nondegeneracy, $[m^*, M] = 0 \Rightarrow m^* = 0$. Hence, we have shown that $\mathcal{R}(H) = 0$.

By hypothesis implies that $MH \subseteq \mathring{e}_\mathcal{R}M$. Now, by Lemma 4, this gives $rH \subseteq \mathring{e}_B B$, and $B$ is left Utumi.

(ii) Assume that $B$ is a right Utumi ring. Let $U^*_\mathcal{R}$ be a submodule of $\mathcal{M}_R$ such that $\perp U^* = 0$. Consider the right ideal $[U^*, M]$ of $B$. If $m[U^*, M] = 0$, then $(m, U^*)M = 0$, hence, since $\mathcal{R}M$ is faithful, $(m, U^*) = 0$, which gives $m = 0$ since $\perp U^* = 0$. Therefore, $1_M([U^*, M]) = 0$, hence, by Lemma 4(i), $\mathcal{L}([U^*, M]) = 0$, which implies that $[U^*, M] \subseteq \mathring{e}_B B$ since $B$ is right Utumi. Then, by Lemma 4(ii), $[U^*, M]M^* \subseteq \mathring{e}_M^*$. But $[U^*, M]M^* \subseteq U^*$, hence $U^*_\mathcal{R} \subseteq \mathring{e}_M^*$, and we have shown that $\perp U^* = 0$ implies $U^*_\mathcal{R} \subseteq \mathring{e}_M^*$. □
Conversely, assume that \( U^* = 0 \Rightarrow UR \subset e M^* \) for any submodule \( UR \) of \( M^* \). Let \( J_B \) be a right ideal of \( B \) such that \( \mathcal{P}(J_B) = 0 \). Then, by Lemma 4(i), \( l_M(J) = 0 \); hence, if \( (m, JM^*) = 0 \), then \( mJ = 0 \) by nondegeneracy, and \( m = 0 \) since \( l_M(J) = 0 \). Thus, \( (JM^*) = 0 \), which, by hypothesis, implies that \( JM^* \subset e M^*_R \). Finally, by Lemma 4(ii), \( JM^* \subset e M^*_R \Rightarrow J_B \subset e B_R \), completing the proof that \( B \) is right Utumi.

**Remarks.** 1. The nondegeneracy condition on \( _RM \) cannot be deleted from the hypothesis of Theorem 7, as we shall see in the following example.

First recall that a CS module is one in which every complement (= essentially closed) submodule is a direct summand, with a ring \( R \) being left or right CS whenever \( _RR \) or \( _RR \) is a CS module. In [1], an example is given of a nonsingular, projective CS module \( P \) whose endomorphism ring, \( B = \text{End} P \), is not left CS (Example 3.3 in [1]). We will show that, for such a \( P \), the condition \( "U^* = 0 \Rightarrow U \subset e P, \) for any submodule \( U \) of \( P \” \) of Theorem 7(i) does hold, and yet \( B = \text{End} P \) is not left Utumi, the reason being that the nondegeneracy condition does not hold in \( P \).

Assume that \( U^* = 0 \) for a submodule \( U \) of \( P \). Then, \( b \in r_B(U) \Rightarrow Ub = 0 \Rightarrow (U, bP^*) = (Ub, P^*) = 0 \Rightarrow bP^* = 0 \) since \( U^* = 0 \), and this last gives \( b = 0 \) since \( bP^* \) is faithful, which shows that \( r_B(U) = 0 \). Now, since \( P \) is a CS module, the essential-closure, \( U^e \), of \( U \) is a direct summand in \( P \), say \( P = U^e \oplus V \), and there is an idempotent \( e \subset B \) such that \( U^eb = 0 \) and \( vb = v \) for \( v \in V \); then \( r_B(U) = 0 \) implies that \( b = 0 \), so \( V = 0 \) and \( U \subset e P \).

To see that \( B \) is not left Utumi, recall first that a ring is left nonsingular, left CS if and only if it is Baer and left Utumi (cf. e.g. [1, Theorem 2.1]); thus, since \( B \) is not left CS, it will suffice to show that \( B \) is Baer: Let \( J \) be any subset of \( B \), then the essential closure, \( (PJ)^e \), of \( PJ \) is a direct summand in \( P \) since \( P \) is CS, say \( P = (PJ)^e \oplus U \); then, letting \( e \) be the idempotent in \( B \) with \( \ker e = (PJ)^e \), we have \( \mathcal{P}(J) = r_B(PJ) = r_B((PJ)^e) = eB \), which proves that \( B \) is a Baer ring.

Finally, to see that nondegeneracy of \( P \) does not hold, we remark that (a) \( P \) nondegenerate \( \Rightarrow I_B(U) \neq 0 \) for every nonzero submodule \( U \) of \( P \), as noted in Remark 1 following Theorem 3; and (b) \( "I_B(U) \neq 0 \) for every \( 0 \neq U \subset P \” \) does not hold in \( P \), because by Lemma 3 of [3] a nonsingular module with this property has a left Utumi endomorphism ring if and only if \( "r_B(U) = 0 \Rightarrow U \subset e P, \) and we have just shown this last to be true in \( P \), whereas \( B \) is not left Utumi.

2. In the special case when the nondegenerate, nonsingular \( _RM \) is \( _RR \), it is easy to see that the conditions in Theorem 7 are precisely the Utumi conditions for a left and right nonsingular \( R \). We verify this for the left Utumi condition, by noting that \( "U^* = 0 \) becomes just \( "r_B(U) = 0 \)” or \( "\mathcal{P}(I) = 0 \)” for \( I \) a left ideal in \( B \). For, in this case, \( B = \text{End}(R) \cong R \); thus, if \( U^* = R I \) is a left ideal in \( R \), then \( I^* = 0 \Rightarrow r_B(I) = 0: b \in r_B(I) \Rightarrow Ib = 0 \Rightarrow (I, bR^*) = (Ib, R^*) = 0 \Rightarrow bR^* = 0 \) since \( I^* = 0 \), \( \Rightarrow b = 0 \) since \( bR^* \) is faithful; and, conversely, \( r_B(I) = 0 \Rightarrow I^* = 0: r^* \in I^* = (I, r^*) = 0 \Rightarrow Ir^* = (I, r^*) = 0 \) for each \( r \in R \), and this last implies that \( r^*r = 0 \) for each \( r \in R \), when we consider \( r^*r \) as being in \( R \cong B \) and use the fact that \( r_B(I) = 0 \); finally, \( r^*R = 0 \Rightarrow r^* = 0 \) since \( R^*_R \) is faithful.
REFERENCES


DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06520