PROPERTIES OF ENDOMORPHISM RINGS
OF MODULES AND THEIR DUALS

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Abstract. Let \( R \mathcal{M} \) be a nonsingular left \( R \)-module whose Morita context is nondegenerate, let \( B = \text{End}_R \mathcal{M} \) and let \( \mathcal{M}^* = \text{Hom}_R(\mathcal{M}, R) \). We show that \( B \) is left (right) strongly modular if and only if any element of \( B \) which has zero kernel in \( R \mathcal{M} \) (\( \mathcal{M} \mathcal{R} \)) has essential image in \( R \mathcal{M} \) (\( \mathcal{M} \mathcal{R} \)), and that \( B \) is a left (right) Utumi ring if and only if any submodule \( \mathcal{R} \mathcal{U} \) of \( R \mathcal{M} \) (\( \mathcal{U} \mathcal{R} \)) such that \( \mathcal{U} \perp = 0 \) (\( \mathcal{U} \mathcal{R}^* = 0 \)) is essential in \( R \mathcal{M} \) (\( \mathcal{M} \mathcal{R} \)).

1. Introduction. Let \( R \mathcal{M} \) be a left \( R \)-module whose standard Morita context is nondegenerate (see Definition 1); let \( B = \text{End}_R \mathcal{M} \) be the ring of \( R \)-endomorphisms of \( R \mathcal{M} \) and let \( \mathcal{M}^* = \text{Hom}_R(\mathcal{M}, R) \) be its dual module. Then \( B \) is left nonsingular if and only if \( R \mathcal{M} \) is nonsingular (i.e. \( \mathcal{M} \) satisfies the following: any \( m \in \mathcal{M} \) with essential annihilator in \( R \mathcal{M} \) must be zero), and \( B \) is right nonsingular if and only if \( \mathcal{M} \mathcal{R} \) satisfies the following condition: If \( \mathcal{U} \mathcal{R} \) is an essential submodule of \( \mathcal{M} \mathcal{R} \) then the annihilator of \( \mathcal{U} \mathcal{R}^* \) in \( B \) must be zero (Proposition 5). This condition certainly holds if \( \mathcal{M} \mathcal{R} \) is nonsingular. Of course, just as for \( R \mathcal{M} \), \( \mathcal{M} \mathcal{R} \) is nonsingular if and only if \( \text{End}_R \mathcal{M}^* \) is right nonsingular. Our concern, however, is with \( B \), which is in general —for example for a nonfinitely generated \( R \mathcal{M} \)—a proper subring of \( \text{End}_R \mathcal{M}^* \); hence a condition on \( R \mathcal{M} \) which is equivalent to a certain left property of \( B \) is not expected to be equivalent to the same right property of \( B \) when it is reflected in \( \mathcal{M} \mathcal{R} \).

In this paper, we investigate this situation and try to pick out some left-right properties of \( B \) which are symmetrically, or almost symmetrically, represented on \( R \mathcal{M} \) and \( \mathcal{M} \mathcal{R} \). For example, we find that \( B \) is left strongly modular if and only if any element of \( B \) which has zero kernel in \( R \mathcal{M} \) has essential image in \( R \mathcal{M} \), while \( B \) is right strongly modular if and only if any element of \( B \) which has zero kernel in \( \mathcal{M} \mathcal{R} \) has essential image in \( \mathcal{M} \mathcal{R} \) (Theorem 3); and we find that \( B \) is a left Utumi ring if and only if every submodule \( \mathcal{R} \mathcal{U} \) of \( R \mathcal{M} \) such that \( \mathcal{U} \perp = 0 \) is essential in \( R \mathcal{M} \), while \( B \) is a right Utumi ring if and only if every submodule \( \mathcal{U} \mathcal{R} \) of \( \mathcal{M} \mathcal{R} \) such that \( \mathcal{U} \mathcal{R}^* = 0 \) is essential in \( \mathcal{M} \mathcal{R} \) (Theorem 7). These conditions naturally raise the general question of how \( B \) sits in \( \text{End}_R \mathcal{M}^* \), a question which we do not treat in this paper, but which we expect to investigate in a future article.
2. Preliminaries. The left and right annihilators in $B$ of a subset $K$ of $B$ will be denoted by $\mathcal{L}(K)$ and $\mathcal{R}(K)$, respectively. The notation $l_M(K), r_M(K), r_B(U), l_B(U^*)$ will be used for the annihilators in $M$ of $K \subseteq B$ in $M^*$ of $K \subseteq B$, in $B$ of $U \subseteq M$ and in $B$ of $U^* \subseteq M^*$, respectively. Also, for $RU \subseteq R_M$ and $U_B \subseteq M^*_B$, we will use: $I_B(U) = \{ b \in B : Mb \subseteq U \}$ and $I_B(U^*) = \{ b \in B : bM^* \subseteq U^* \}$. The notation $\rho U \subseteq \ell_R M$ will be used to indicate that $U$ is an essential $R$-submodule of $M$, i.e. $U$ intersects nontrivially every nonzero $R$-submodule of $M$. Recall that $\rho M$ is said to be nonsingular in case, for $m \in M$, $l_R(m) \subseteq \ell_R R \Rightarrow m = 0$; $B$ is said to be left (right) nonsingular if $\rho B (B_B)$ is nonsingular.

We recall the following definition and proposition from [4]:

**Definition 1.** Let $(R, M, N, S)$ be a Morita context; that is, let $R_M$ and $S_N$ be bimodules with an $R$-$R$ bimodule homomorphism $(.,.) : M \otimes S N \rightarrow R$ and an $S$-$S$ bimodule homomorphism $[ , ] : N \otimes R M \rightarrow S$ satisfying

\[
mx[nx,m_2] = (mx,nx)m_2 \quad \text{and} \quad nx(mx,n_2) = [n_1,m_1]n_2
\]

for all $m_1, m_2 \in M$ and $n_1, n_2 \in N$.

Then $(R, M, N, S)$ is said to be nondegenerate if and only if the four modules $R_M$, $M_S$, $S_N$, $N_R$ and the two pairings are faithful (the latter meaning that $(m, N) = 0$ implies $m = 0$, and three analogous implications).

This is equivalent to the eight natural maps associated with the Morita context being injective (for example, two of these maps are: $m \rightarrow (m, -)$ and $r \rightarrow (n \rightarrow nr) \in \text{End}(S_N)$, for $m \in M$, $n \in N$ and $r \in R$). The standard context $(R, M, M^*, B)$ of a module $R_M$ is nondegenerate if and only if $R_M$ is torsionless and faithful and the right annihilator of trace $(\rho_M)$ is zero. We shall call such a module—i.e. one whose standard context is nondegenerate—a nondegenerate module, for brevity.

**Proposition 1 [4, Proposition 14].** If the context $(R, M, N, S)$ is nondegenerate, and if one of $R_M, R_M, S_N, S_S$ is nonsingular, then all of them are nonsingular.

Henceforth, unless otherwise indicated, let $R_M$ be a nondegenerate, nonsingular left $R$-module. Then, by the preceding, $R_M, M_B, M_M^*, M_B^*$ and the two pairings are faithful, and $R_R, R_M^*$ and $B_B$ are nonsingular. $(.,.)$ and $[.,.]$ will denote the pairings associated with the standard context for $R_M$, i.e. $(.,.)$ is defined by $(m,f) = mf$ for $m \in M$ and $f \in M^*$, and $[f,m]$ is defined by $m_1[f,m] = (m_1,f)m$ for all $m, m_1 \in M$ and $f \in M^*$.

If $RU$ is a submodule of $R_M$ then $[M^*,U]$ indicates the left ideal of $B$: $[M^*,U] = \{ \sum_{i=1}^n [m_i^*,u_i] : m_i^* \in M^*, u_i \in U \}$, and similarly for $[U^*,M]$ where $U^*_R$ is a submodule of $M^*_R$. Also, $U_1 = \{ m^* \in M^*: (U,m^*) = 0 \}$ and $U^*_1 = \{ m \in M: (m,u^*) = 0 \}$.

The well-known fact that, for a nonsingular module $R_M$, any $R$-homomorphism $f$, to $R_M$ from any other $R$-module, which has essential kernel is zero, will be used repeatedly without comment.

The following lemma will be useful to us in the sequel.

**Lemma 2.** For $K \subseteq B$, $\mathcal{L}(K) = I_B|_M(K) = I_B(KM^*)$, and $\mathcal{R}(K) = r_B(MK) = I^*_B|_M(K)$. 

Proof.

(b ∈ \mathcal{L}(K) ⇔ bK = 0 ⇔ MbK = 0 ⇔ Mb \subseteq l_M(K) ⇔ b \in l_Bl_M(K);

(b ∈ \mathcal{L}(K) ⇔ bK = 0 ⇔ bKM* = 0 ⇔ b \in l_B(KM*);

(b ∈ \mathcal{R}(K) ⇔ Kb = 0 ⇔ MKb = 0 ⇔ b \in r_B(MK);

(b ∈ \mathcal{R}(K) ⇔ Kb = 0 ⇔ KbM* = 0 ⇔ bM* \subseteq r_M*(K) ⇔ b \in l_Br_M*(K). \quad \square

3. Strongly modular and Utumi endomorphism rings. In [2], a Baer *-ring \( B \) is called strongly modular in case, for all \( b \in B \), \( \mathcal{R}(b) = 0 \) implies that \( bB \) is essential in \( B \). Because of the involution, the definition is left-right symmetric. In the absence of an involution, call a ring \( B \) left strongly modular if, for \( b \in B \), \( \mathcal{L}(b) = 0 \Rightarrow bB \subseteq \epsilon_B B \), and right strongly modular if \( \mathcal{R}(b) = 0 \Rightarrow bB \subseteq \epsilon_B B \). It turns out that the properties of left and right strong modularity of \( B = \text{End}_R M \) are equivalent to almost symmetric conditions on \( _RM \) and \( _MR \).

Theorem 3. (i) \( B \) is left strongly modular if and only if, for each \( b \in B \), \( l_M(b) = 0 \Rightarrow Mb \subseteq \epsilon_R M \);

(ii) \( B \) is right strongly modular if and only if, for each \( b \in B \), \( r_M*(b) = 0 \Rightarrow bM* \subseteq \epsilon_M R * \).

Proof. By comparing the definition of left strong modularity with the condition on \( _RM \) in (i), it is easily seen that (i) will follow as soon as we show that “\( \mathcal{L}(b) = 0 \)” is equivalent to “\( l_M(b) = 0 \)” and that “\( \mathcal{R}(b) = 0 \)” is equivalent to “\( r_M*(b) = 0 \)” and that “\( bB \subseteq \epsilon_B B \)” is equivalent to “\( bM* \subseteq \epsilon_M R * \)”.

Lemma 4. (i) For any subset \( K \) of \( B \), \( \mathcal{L}(K) = 0 \) if and only if \( l_M(K) = 0 \) and \( \mathcal{R}(K) = 0 \) if and only if \( r_M*(K) = 0 \).

(ii) For any left ideal \( BH \) of \( B \), \( BH \subseteq \epsilon_B B \) if and only if \( MH \subseteq \epsilon_R M \); and for any right ideal \( JB \) of \( B \), \( JB \subseteq \epsilon B \) if and only if \( JM* \subseteq \epsilon_M R * \).

Proof. (i) Let \( K \subseteq B \) and consider the submodule \( l_M(K) \) of \( _RM \). If \( l_M(K) \neq 0 \), let \( 0 \neq m \in l_M(K) \); then, by nondegeneracy, there is \( m* \in M* \) such that \( [m*, m] \neq 0 \). Then, since \( M_B \) is faithful, \( 0 \neq M[m*, m] = (M, m*)m \subseteq Rm \subseteq l_M(K) \); that is, \( 0 \neq [m*, m] \in l_Bl_M(K) \). Hence, \( l_M(K) = 0 \) if and only if \( \mathcal{L}(K) = l_Bl_M(K) = 0 \).

Similarly, if \( 0 \neq m* \in r_M*(K) \), then nondegeneracy gives \( m \in M \) such that \( [m*, m] \neq 0 \), and since \( B_M \) is faithful, \( [m*, m] \) is a nonzero element of \( l_Bl_M(K) \); hence \( r_M*(K) = 0 \) if and only if \( \mathcal{R}(K) = l_Bl_M(K) = 0 \).

(ii) Let \( BH \) be an essential left ideal of \( B \) and let \( 0 \neq m \in M \). Then \( [M*, m] \cap H \neq 0 \), and, since \( M_B \) is faithful,

\( 0 \neq M([M*, m] \cap H) \subseteq M [M*, m] \cap MH = (M, M*)m \cap MH \subseteq Rm \cap MH \),

proving that \( MH \subseteq \epsilon_R M \).

Conversely, assume that \( MH \subseteq \epsilon_R M \) for some left ideal \( BH \) of \( B \) and let \( 0 \neq c \in B \). Then, since \( M_B \) is faithful, \( MC \neq 0 \), and hence \( MC \cap MH \neq 0 \). By nondegeneracy,

\( 0 \neq [M*, MC \cap MH] \subseteq [M*, M] \cap [M*, MH] \subseteq Bc \cap [M*, MH] \).
This shows that \([M^*, MH] \subseteq e_B H\), and hence, since \([M^*, MH] \subseteq e_B H\), we have \(\mu_H \subseteq e_B B\).

Similarly, if \(J\) is an essential right ideal of \(B\) and \(0 \neq m^* \in M^*\), then, by nondegeneracy, \([m^*, M] \neq 0\), hence \([m^*, M] \cap J \neq 0\). Since \(M^*_B\) is faithful, this implies

\[
0 \neq ([m^*, M] \cap J) M^* \subseteq [m^*, M] M^* \cap JM^* = m^*(M, M^*) \cap JM^* \subseteq m^* R \cap JM^*;
\]

so \(JM^* \subseteq e_M R^*\).

Conversely, if \(JM^* \subseteq e_M R^*\) for some right ideal \(J_B\) of \(B\), and \(0 \neq c \in B\), then \(JM^* \cap cM^* \neq 0\) and \([JM^* \cap cM^*, M] \neq 0\) by nondegeneracy; hence,

\[
0 \neq ([JM^* \cap cM^*, M] \subseteq [JM^*, M] \cap [cM^*, M] \subseteq [JM^*, M] \cap cB.
\]

This implies that \([JM^*, M] \subseteq e_B B\), and hence, since \([JM^*, M] \subseteq J\), we have \(J_B \subseteq e_B B\). □

**Remarks.**

1. One property of nondegenerate modules that can be deduced from the proof of Lemma 4 is that \(I_B(U) = 0\) if and only if \(U = 0\) for a submodule \(RU\) of \(R M\), and similarly for \(U_R^* \subseteq M_R^*\).

2. In the proof of Lemma 4(ii), we have shown that \(rU \subseteq e_R M = [M^*, U] \subseteq e_B B\) and \(U_R^* \subseteq e_M R^* \Rightarrow [U^*, M] \subseteq e_B B\).

Aside from completing the proof of Theorem 3, Lemma 4 is also useful in giving a condition on \(M^*_R\) which is equivalent to right nonsingularity of \(B\), as in the next result.

**Proposition 5.** \(B\) is right nonsingular if and only if, for any submodule \(U_R^* \subseteq M^*_R\), \(U_R^* \subseteq e M_R^* \Rightarrow I_B(U^*) = 0\).

**Proof.** It was shown in [3, Proposition 1] that, under our present hypotheses, \(B\) is right nonsingular if and only if, for any submodule \(RU \subseteq R M\), \(r_B(U) \subseteq e_B B\Rightarrow U = 0\).

Assume that \(B\) is right nonsingular and suppose that \(U_R^* \subseteq e M_R^*\); then, as noted in Remark 2 above, \([U^*, M] \subseteq e_B B\). We have \((Ml_B(U^*))[U^*, M] = (Ml_B(U^*)][U^*, M] = 0\); therefore \([U^*, M] \subseteq r_B(Ml_B(U^*))\), which implies \(r_B(Ml_B(U^*)) \subseteq e_B B\). Hence, by [3, Proposition 1], since \(B\) is right nonsingular, this implies that \(Ml_B(U^*) = 0\); hence, since \(M_R^*\) is faithful, we have \(I_B(U^*) = 0\).

Conversely, assume that \(U_R^* \subseteq e M_R^*\) implies \(I_B(U^*) = 0\). Suppose that \(RU \subseteq R M\) such that \(r_B(U) \subseteq e_B B\). Then, by Lemma 4(ii), \(U_R^* = r_B(U) M^* \subseteq e M_R^*\). Hence, by hypothesis, \(I_B(r_B(U) M^*) = 0\). But \(I_B(U) \subseteq I_B(r_B(U) M^*)\) since, always, \(I_B(U) r_B(U) = 0\); hence \(I_B(U) = 0\), which, by nondegeneracy (see Remark 1), implies that \(U = 0\), completing the proof. □

A ring \(B\) is said to be a left Utumi ring in case, for any left ideal \(H\) of \(B\), \(\mathcal{S}(H) = 0 \Rightarrow \mu_H \subseteq e_B B\); \(B\) is called a right Utumi ring if, for any right ideal \(J_B\) of \(B\), \(\mathcal{S}(J_B) = 0 \Rightarrow J_B \subseteq e_B B\). In [2], it is shown that a strongly modular Baer \(*\)-ring is left and right Utumi [2, Theorem 2.3]. In our situation, i.e. for \(B = \text{End}_R M\), where \(R M\) is nondegenerate and nonsingular, it is easily shown that a left and right strongly modular Baer ring satisfies the Utumi conditions for principal left and right
ideals. In fact, $B$ need not be a Baer ring to show this; rather, left and right nonsingularity of $B$ is sufficient, with left and right strong modularity, in order to obtain the Utumi conditions for principal ideals, as can be seen in Proposition 6. For the full Utumi conditions on $B$, however, we can give, in Theorem 7, conditions on $R M$ and $M_R^*$ which appear quite symmetrical.

**Proposition 6.** If $R M$ is such that $B = \text{End}_R M$ is a left and right nonsingular, left and right strongly modular ring, then

(i) $\mathcal{L}(bB) = 0 \Rightarrow bB \subset \epsilon B_B$, and (ii) $\mathcal{R}(Bb) = 0 \Rightarrow Bb \subset \epsilon_B B$.

**Proof.** (i)

\[ \mathcal{L}(bB) = \mathcal{L}(b) = 0 \Rightarrow bB \subset \epsilon B_B \quad \text{since} \quad B \text{ is left strongly modular,} \]

\[ \Rightarrow \mathcal{R}(Bb) = 0 \quad \text{since} \quad B \text{ is left nonsingular,} \]

\[ \Rightarrow bB \subset \epsilon B_B \quad \text{since} \quad B \text{ is right strongly modular.} \]

(ii)

\[ \mathcal{R}(Bb) = \mathcal{R}(b) = 0 \Rightarrow Bb \subset \epsilon_B B \quad \text{since} \quad B \text{ is right strongly modular,} \]

\[ \Rightarrow \mathcal{L}(bB) = 0 \quad \text{since} \quad B \text{ is right nonsingular,} \]

\[ \Rightarrow Bb \subset \epsilon_B B \quad \text{since} \quad B \text{ is left strongly modular}. \]

For our last result, $R M$ is assumed to satisfy the standing hypothesis, i.e. $R M$ is nonsingular and nondegenerate.

**Theorem 7.** (i) $B = \text{End}_R M$ is a left Utumi ring if and only if, for any submodule $R U$ of $R M$, $U^\perp = 0 \Rightarrow R U \subset \epsilon_R M$; and (ii) $B = \text{End}_R M$ is a right Utumi ring if and only if, for any submodule $U_R^*$ of $M_R^*$, $U^* = 0 \Rightarrow U_R^* \subset \epsilon_{M_R^*}$.

**Proof.** (i) Assume that $B$ is a left Utumi ring; then, by [3, Lemma 3], we have, for any submodule $R X$ of $R M$, $r_B(R X) = 0 \Rightarrow R X \subset \epsilon_R M$. Let $R U$ be a submodule of $R M$ such that $U^\perp = 0$. Then $b \in r_B(U) \Rightarrow Ub = 0 \Rightarrow (U, bm^*) = (Ub, m^*) = 0$, for each $m^* \in M^*$, $\Rightarrow bm^* = 0$ since $U^\perp = 0$; but this means $bM^* = 0$, therefore, since $R M$ is faithful, $b = 0$. Hence $r_B(U) = 0$, which implies, since $B$ is left Utumi, that $R U \subset \epsilon_R M$.

Conversely, assume that $R U^\perp = 0 \Rightarrow R U \subset \epsilon_R M$ for every $R U \subset R M$. Let $R H$ be a left ideal of $B$ with $\mathcal{R}(R H) = 0$. If $(M H, m^*) = 0$, then $(M, H m^*) = 0$, which implies $H m^* = 0$ by nondegeneracy. Then $[H m^*, M] = 0$, i.e. $H[m^*, M] = 0$, which implies $[m^*, M] = 0$ since $\mathcal{R}(H) = 0$. Again by nondegeneracy, $[m^*, M] = 0 \Rightarrow m^* = 0$. Hence, we have shown that $(M H, m^*) = 0 \Rightarrow m^* = 0$, i.e. $(M H)^\perp = 0$, which by hypothesis implies that $M H \subset \epsilon_R M$. Now, by Lemma 4(ii), this gives $R H \subset \epsilon_B B$, and $B$ is left Utumi.

(ii) Assume that $B$ is a right Utumi ring. Let $U_R^*$ be a submodule of $M_R^*$ such that $U^* = 0$. Consider the right ideal $[U^*, M]$ of $B$. If $m[U^*, M] = 0$, then $(m, U^*) M = 0$, hence, since $R M$ is faithful, $(m, U^*) = 0$, which gives $m = 0$ since $U^* = 0$. Therefore, $I_M([U^*, M]) = 0$, hence, by Lemma 4(i), $\mathcal{L}([U^*, M]) = 0$, which implies that $[U^*, M] \subset \epsilon_B B$ since $B$ is right Utumi. Then, by Lemma 4(ii), $[U^*, M] M^* \subset \epsilon_{M_R^*}$. But $[U^*, M] M^* \subset U^*$, hence $U_R^* \subset \epsilon_{M_R^*}$, and we have shown that $U^* = 0$ implies $U_R^* \subset \epsilon_{M_R^*}$.
Conversely, assume that \( \perp U^* = 0 \Rightarrow U_R^* \subseteq \mathcal{M}_R^* \) for any submodule \( U_R^* \) of \( \mathcal{M}_R^* \). Let \( J_B \) be a right ideal of \( B \) such that \( \mathcal{P}(J_B) = 0 \). Then, by Lemma 4(i), \( l_M(J) = 0 \); hence, if \( (m, JM^*) = 0 \), then \( mJ = 0 \) by nondegeneracy, and \( m = 0 \) since \( l_M(J) = 0 \). Thus, \( \perp (JM^*) = 0 \), which, by hypothesis, implies that \( JM^* \subseteq \mathcal{M}_R^* \). Finally, by Lemma 4(ii), \( JM^* \subseteq \mathcal{M}_R^* \Rightarrow J_B \subseteq \mathcal{B}_B^* \), completing the proof that \( B \) is right Utumi.

**Remarks.** 1. The nondegeneracy condition on \( \mathcal{R} \mathcal{M} \) cannot be deleted from the hypothesis of Theorem 7, as we shall see in the following example.

First recall that a CS module is one in which every complement (= essentially closed) submodule is a direct summand, with a ring \( R \) being left or right CS whenever \( _RR \) or \( _RR \) is a CS module. In [1], an example is given of a nonsingular, projective CS module \( P \) whose endomorphism ring, \( B = \text{End} \mathcal{P} \), is not left CS (Example 3.3 in [1]). We will show that, for such a \( P \), the condition \( \perp U = 0 \Rightarrow U \subseteq \mathcal{P} \) for any submodule \( U \) of \( \mathcal{P} \) of Theorem 7(i) does hold, and yet \( B = \text{End} \mathcal{P} \) is not left Utumi; the reason being that the nondegeneracy condition does not hold in \( P \).

Assume that \( U \perp = 0 \) for a submodule \( U \) of \( P \). Then, \( b \in r_B(U) \Rightarrow Ub = 0 \Rightarrow (U, bP^*) = (Ub, P^*) = 0 \Rightarrow bP^* = 0 \) since \( U \perp = 0 \), and this last gives \( b = 0 \) since \( BP^* \) is faithful, which shows that \( r_B(U) = 0 \). Now, since \( P \) is a CS module, the essential-closure, \( U^e \), of \( U \) is a direct summand in \( P \), say \( P = U^e \oplus V \), and there is an idempotent \( b \in B \) such that \( U^eb = 0 \) and \( vb = v \) for \( v \in V \); then \( r_B(U) = 0 \) implies that \( b = 0 \), so \( U = 0 \) and \( U \subseteq \mathcal{P} \).

To see that \( B \) is not left Utumi, recall first that a ring is left nonsingular, left CS if and only if it is Baer and left Utumi (cf. e.g. [1, Theorem 2.1]); thus, since \( B \) is not left CS, it will suffice to show that \( B \) is Baer: Let \( J \) be any subset of \( B \), then the essential-closure, \( (PJ)^e \), of \( PJ \) is a direct summand in \( P \) since \( P \) is CS, say \( P = (PJ)^e \oplus U \); then, letting \( e \) be the idempotent in \( B \) with \( \text{ker} \mathcal{E} = (PJ)^e \), we have \( r_B(J) = r_B(PJ) = r_B((PJ)^e) = eB \), which proves that \( B \) is a Baer ring.

Finally, to see that nondegeneracy of \( P \) does not hold, we remark that (a) \( P \) nondegenerate \( \Rightarrow I_B(U) \neq 0 \) for every nonzero submodule \( U \) of \( P \), as noted in Remark 1 following Theorem 3; and (b) \( \text{”} I_B(U) \neq 0 \text{” for every } 0 \neq U \subseteq P \) does not hold in \( P \), because by Lemma 3 of [3] a nonsingular module with this property has a left Utumi endomorphism ring if and only if \( \text{”} r_B(U) = 0 \Rightarrow U \subseteq \mathcal{P} \text{”} \), and we have just shown this last to be true in \( P \), whereas \( B \) is not left Utumi.

2. In the special case when the nondegenerate, nonsingular \( _RR \mathcal{M} \) is \( _RR \), it is easy to see that the conditions in Theorem 7 are precisely the Utumi conditions for a left and right nonsingular \( R \). We verify this for the left Utumi condition, by noting that \( \perp = 0 \) becomes just \( r_B(U) = 0 \) or \( r(\mathcal{I}) = 0 \) for \( \mathcal{I} \) a left ideal in \( B \). For, in this case, \( B = \text{End}(RR) \cong R \); thus, if \( ru = _RR \mathcal{I} \) is a left ideal in \( R \), then \( I \perp = 0 \Rightarrow r_B(I) = 0 \Rightarrow b \in r_B(I) \Rightarrow b1 = 0 \Rightarrow (I, bR^*) = (bR^*) = 0 \Rightarrow bR^* = 0 \) since \( I \perp = 0 \), \( b = 0 \) since \( R^* \) is faithful; and, conversely, \( r_B(I) = 0 \Rightarrow I \perp = 0 \Rightarrow r \in I \perp = 0 \Rightarrow 1r = 0 \Rightarrow (1, r^*) = 0 \) for each \( r \in R \), and this last implies that \( r^* = 0 \) for each \( r \in R \), when we consider \( r^* \) as being in \( R \cong B \) and use the fact that \( r_B(I) = 0 \); finally, \( r^* \mathcal{R} = 0 \Rightarrow r^* = 0 \) since \( R_R^* \) is faithful.
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