THE ESSENTIAL BOUNDARY OF CERTAIN SETS

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ABSTRACT. The essential boundary of a measurable set is related to the de Giorgi perimeter and was introduced by Vol'pert in his "improvement" of Federer's work.

For a totally disconnected compact set of positive measure in n space the essential boundary can be of Hausdorff n — 1 dimension but cannot have σ finite (n — 1)-measure.

Let E ⊂ R^n be a measurable set with respect to Lebesgue measure. It is said to be of finite perimeter if all the partial derivatives μ_1(A),...,μ_n(A) of its characteristic function χ are totally finite measures over the Borel sets A ⊂ R^n. It is well known that μ_i = μ_i(R^n) = ∫ v_i, where v_i is the infimum of the variations in χ_i of all functions equivalent to χ and the integration is over the (n — 1)-space orthogonal to the χ_i axis, Oχ_i. The value of the perimeter is then the variation measure of the vector valued measure (μ_1(A),...,μ_n(A)), evaluated for R^n.

It was shown by Federer [1] that the perimeter is equal to the (n — 1)-measure of a set that he called the reduced boundary of E, consisting of those points at which a certain generalized normal exists. Specifically, a point p is in the reduced boundary of E if there is an (n - 1)-plane π through p such that the part of E on one side of π has density 0 at p, and the part of CE on the other side of π has density 0 at p. (The k-measure λ_k(E) of a set in R^n will mean the k-dimensional Hausdorff measure, normalized so that the k-dimensional unit cube has measure 1.) Vol'pert [2] showed that in the result the reduced boundary may be replaced by the essential boundary δ_e(E), consisting of those points of R^n which are neither points of density 1 nor points of density 0 of E. Clearly, the essential boundary of E contains the restricted boundary. Vol'pert's remarkable theorem asserts that, if E has finite perimeter, then the Hausdorff (n — 1)-measure of the restricted and essential boundaries are equal. The Lebesgue theorem guarantees that δ_e(E) is of n-measure zero, but in its perimeter role δ_e(E) has more of an (n — 1)-dimensional flavor, and evidently coincides with the ordinary boundary ∂(E) when this is a sufficiently smooth surface.

Going to the opposite extreme, in this note we shall discuss the possible nature of δ_e(E), with respect to (n — 1)-measure, when E is a nowhere dense set of positive measure in R^n.
In the case \( n = 1 \), it is easy to see that \( \partial_a(E) \) is then of non-\( \sigma \)-finite 0-measure; in other words, it is an uncountable set. In fact, since there exists an interval in which the relative measure of \( E \) is greater than \( \frac{1}{2} \), while intervals disjoint from \( E \) are dense in \( R \), by continuity we can find two disjoint compact intervals \( I(0), I(1) \) such that

\[
0 < \lambda_1(E \cap I(0)) = \lambda_1(E \cap I(1)) = \frac{1}{2}\lambda_1(I(0)) = \frac{1}{2}\lambda_1(I(1)) < \frac{1}{4}.
\]

Similarly, inside each \( I(i) \), \( i = 0, 1 \), we can find two disjoint compact intervals \( I(i, 0), I(i, 1) \) of length less than \( \frac{1}{4} \), in each of which the relative measure of \( E \) is again \( \frac{1}{4} \), and so on. Now \( \partial_a(E) \) evidently contains the uncountable set

\[
\bigcap_{r=1}^{\infty} \bigcup \{ I(\epsilon_1, \ldots, \epsilon_r) : \epsilon_r = 0 \text{ or } 1 \}.
\]

In \( R^n \), \( n \geq 2 \), a nowhere dense closed set may be of positive \( n \)-measure and yet have finite perimeter. Indeed, let \( a_0 \) denote the open unit ball, and let \( a_1, a_2, \ldots \) be a sequence of disjoint open balls in \( a_0 \) such that \( \bigcup_{m \geq 1} a_m \) is dense in \( a_0 \), but of smaller \( n \)-measure than \( a_0 \), and \( \sum_{m \geq 1} c_m < \infty \), where \( c_m \) is the circumference of \( a_m \). Then \( F = a_0 \setminus \bigcup_{m \geq 1} a_m \) is a nowhere dense closed set of positive \( n \)-measure; however, it is clear that for the characteristic function of \( F \) we have \( \mu_i \leq \sum_{m \geq 0} c_m \) for \( i = 1, \ldots, n \), so that \( F \) has finite perimeter. It can of course be established directly that the \((n - 1)\)-measure of \( \partial_a(F) \) is finite but this is appreciably more difficult.

As can be seen from Theorem 3 below, it is possible for a set such as \( F \) to have finite perimeter only because (although nowhere dense) it is far from being totally disconnected; indeed, in a certain sense, for most straight lines that intersect \( F \) in positive 1-measure, the intersection must consist largely of complete intervals. In fact, we shall show that if \( F \) is any closed set, then on almost every straight line in any given direction the essential boundary \( \partial_a(F) \) includes an important part of the topological boundary of \( F \), relative to the line. Consequently, when \( F \) is totally disconnected the set \( \partial_a(F) \) must be of non-\( \sigma \)-finite \((n - 1)\)-measure.

We first point out that in this case, \( \partial_a(F) \) is not necessarily of Hausdorff dimension greater than \( n - 1 \).

**Theorem 1.** In \( R^n \) there exists a totally disconnected compact set \( F \) of positive \( n \)-measure for which \( \partial_a(F) \) is of Hausdorff dimension \( n - 1 \).

**Proof.** Let \( C \) denote the \( n \)-dimensional unit cube \([0,1]^n\). Let \( \theta_1, \theta_2, \ldots \) be a sequence of positive numbers such that the series \( \sum_{r \geq 2}((2r^2 + 1)\theta_r)^n \) converges for every \( \alpha > 0 \) and \( \sum_r \theta_r < 1/\alpha \); for example, let \( \theta_r = 1/n(2r^2 + 1)^2 \).

Let \( q_1, q_2, \ldots \) be an enumeration of all rational numbers in \([0,1]\). Let \( I \), denote the open interval \((a_r, b_r)\) with center at \( q_r \) and of length \( \theta_r \). Let \( S_1 \) denote the slab \( I_r \times [0,1] \times \cdots \times [0,1] \), and define \( S_i \) similarly for \( i = 2, \ldots, n \) as the product of the interval \( I_r \), placed on the axis \( Ox_r \) with a unit cube in the \((n-1)\)-space orthogonal to \( Ox_r \); let \( S_r = S_{1r} \cup \cdots \cup S_{nr} \) and \( F = C \setminus \bigcup_r S_r \). Then \( F \) is a totally disconnected compact set with

\[
\lambda_n(F) \geq \lambda_n(C) - \sum_r \lambda_n(S_r \cap C) \geq 1 - n \sum_r \theta_r > 0.
\]
Now let $T_{ri}$ denote the slab consisting of all points at a distance less than $r^2 \theta_i$ from $S_{ri}$, put $T_r = T_{r1} \cup \cdots \cup T_{rn}$, and let $H = \limsup(T_r \cap C)$. Then $H$ is of Hausdorff dimension $n - 1$, because for every $R$ we have $H \subset \bigcup_{r > R} (T_r \cap C)$, for each $r$, $i$ the set $T_{ri} \cap C$ can be covered by $O([1/(2r^2 + 1)\theta_i]^{n-1})$ cubes of side $(2r^2 + 1)\theta_i$, and for every $\beta > n - 1$,
\[
\sum_{r > R} \left( \frac{1}{(2r^2 + 1)\theta_i} \right)^n (2r^2 + 1)\theta_i \beta \to 0 \quad \text{as } R \to \infty.
\]

It will follow from Theorem 3 that $\partial_e(F)$ has non-$\sigma$-finite $(n - 1)$-measure, so it is now enough to show that $\partial_e(F) \subset H$, and hence enough to show that if $x \in F \setminus H$ then the density of $\bigcup_r S_r$ at $x$ is zero. Let $x = (x_1, \ldots, x_n) \in F \setminus H$. It is enough to show, for example, that the density of $\bigcup_r S_r$ at $x$ in the orthant \{ $y$: $y_i > x_i$ for $i = 1, \ldots, n$ \} is zero.

Given $\varepsilon > 0$, choose $R$ so large that $\sum_{r > R} 1/r^2 < \varepsilon/n$. Since $x \notin H$, we can choose $Q$ so that $x \notin \bigcup_{r > Q} T_r$, and we may suppose $Q > R$. Therefore,
\[
\sum_{r > Q} \frac{1}{r^2} < \frac{\varepsilon}{n}.
\]

We can choose $\delta > 0$ so small that $\bigcup_{r = 1}^{Q-1} S_r$ does not meet the cube $C_\delta = \{ y: 0 \leq y_i < x_i < \delta \text{ for } i = 1, \ldots, n \}$.

Given $0 < h < \delta$, we shall show (establishing the conclusion) that
\[
\lambda_n \left[ C_h \cap \bigcup_r S_r \right] \leq \varepsilon \lambda_n (C_h).
\]

Consider any slab $S_{ri}$ such that $S_{ri} \cap C_h \neq \emptyset$; then $r \geq Q$. Without loss of generality we may suppose that $i = 1$. Because $x \notin T_{r1}$, but $S_{r1} \cap C \neq \emptyset$, we have $x_1 < a_r - r^2 \theta_r < a_r < x_1 + h$, and therefore
\[
\lambda_1 \left[ (x_1, x_1 + h) \cap I_r \right] \leq \lambda_1 (I_r) = \theta_r = (1/r^2) \left[ a_r - (a_r - r^2 \theta_r) \right] < (1/r^2) h,
\]
from which it follows that $\lambda_n (C_h \cap S_{ri}) \leq (1/r^2) \lambda_n (C_h)$. By (1) this implies the required result (2).

We proceed to show that for a totally disconnected compact set $K$ of positive $n$-measure the set $\partial_e(K)$ is of non-$\sigma$-finite $(n - 1)$-measure. We first obtain a result on arbitrary measurable sets. Let $E \subset R^n$ be measurable with $\lambda_n (E) > 0$. We adjust $E$ in the $x_1$ direction as follows: on every open interval in any line parallel to $Ox_1$, if $E$ has zero linear measure in the interval, transfer the entire interval to $CE$, and if $CE$ has linear measure zero in the interval, place the entire interval in $E$. By the linear density theorem, this adjustment only changes $E$ by a set of $n$-measure zero. Such an adjusted set may be called $x_1$-smooth. Thus, if $E$ is $x_1$-smooth, then for every open interval $I$ situated in a line parallel to $Ox_1$ we have that $\lambda_1 (E \cap I) = 0$ implies $E \cap I = \emptyset$ and $\lambda_1 (CE \cap I) = 0$ implies $(CE) \cap I = \emptyset$.

In what follows it will be convenient to regard a typical point in $R^n$ as $(x; y)$ where $x \in R$ and $y \in R^{n-1}$. For any set $E \subset R^n$ and any $y \in R^{n-1}$, by $\partial_e(E \cdot y)$ we shall mean the ordinary boundary in $R$ of the section $E \cdot y = \{ x: (x; y) \in E \}$, that
is, \( \partial(E^y) = \overline{E^y} \cap (R \setminus E^y) \). By \( d(E, p) \) and \( d(E, p) \), where \( E \subset R^n \) and \( p \in R^n \), we mean the respective limits as \( \delta \to 0^+ \) of the infimum and supremum of \( \{ \lambda_n(E \cap S)/\lambda_n(S) : S \in \mathcal{S}(\delta, p) \} \) where \( \mathcal{S}(\delta, p) \) is the collection of all open cubes with sides parallel to the axes containing \( p \) and with side of length less than \( \delta \). A subset \( A \) of a set \( B \) is said to be residual in \( B \) if \( B \setminus A \) is the union of countably many sets each of which is nowhere dense in \( B \).

**Theorem 2.** If \( E \) is any \( x_1 \)-smooth measurable set in \( R^n \), then for almost all \( y \in R^{n-1} \) the set

\[
\{ x : d(E, (x; y)) = 0 \text{ and } d(E, (x; y)) = 1 \} \cap (E^y)
\]

is residual in \( \partial(E^y) \).

**Proof.** It is sufficient to prove that for almost all \( y \in R^{n-1} \) the set \( \{ x : d(E, (x; y)) = 0 \} \cap \partial(E^y) \) is residual in \( \partial(E^y) \), because this same result applied to \( CE \) will show that \( \{ x : d(CE, (x; y)) = 0 \} \cap \partial(CE)^y \) is also residual in \( \partial((CE)^y) = \partial(E^y) \) for almost all \( y \in R^{n-1} \).

Given any positive integer \( r \), let \( A(r) \) denote the union of all open cubes \( C \) such that \( \lambda_n(C) < r^{-n} \) and \( \lambda_n(E \cap C) < r^{-1}\lambda_n(C) \). It will be enough to show that for almost all \( y \in R^{n-1} \) the set \( A(r)^y \cap \partial(E^y) \) is dense in \( \partial(E^y) \), since \( A(r) \) is open and therefore \( \partial(E^y) \setminus A(r)^y \) will then be nowhere dense for each \( r \), and consequently \( \{ x : d(E, (x; y)) > 0 \} \cap \partial(E^y) \) is of the first category in \( \partial(E^y) \).

We henceforth regard \( r \) as fixed and write \( A \) for \( A(r) \). Suppose, if possible, that for a set of \( y \in R^{n-1} \) of positive outer \((n-1)\)-measure the set \( A^y \cap \partial(E^y) \) is not dense in \( \partial(E^y) \); we shall derive a contradiction. For each such \( y \) there exists a nonempty open interval \( I \subset R \) with rational endpoints such that

\[
I \cap \partial(E^y) \neq \emptyset \quad \text{and} \quad \partial(E^y) \cap A^y = \emptyset.
\]

Hence there exists a set \( Z \subset R^{n-1} \) of positive outer \((n-1)\)-measure and a fixed nonempty open interval \( I \subset R \) such that \((3)\) holds for all \( y \in Z \). Whenever \((3)\) holds, we have \( I \cap (CE)^y \neq \emptyset \) and therefore \( \lambda_n^*[I \cap ((CE)^y)] > 0 \), because \( E \) is \( x_1 \)-smooth. Since almost all points of \( CE \) are points of density 1 for \( CE \), it follows that for almost every \( z \in Z \) there exists \( x \in I \) for which \( d(E, (x; z)) = 0 \). Choose any such point \( z \in Z \) which is also a point of outer \((n-1)\)-density 1 for \( Z \), choose \( h > 0 \) so small that \( \lambda_{n-1}(Z \cap K) > (1 - \frac{1}{2}r^{-1})\lambda_{n-1}(K) \) for every open cube \( K \subset R^{n-1} \) of side less than \( h \), containing \( z \), and let \( x_0 \in I \) be such that \( d(E, (x_0; z)) = 0 \).

Let \( C_0 \) be an open cube of side less than \( \min(h, r^{-1}) \) with \( (x_0; z) \in C_0 \), and

\[
\lambda_n(E \cap C_0) < \frac{1}{2}r^{-1}\lambda_n(C_0),
\]

and \( C_0 = J_0 \times K \) where \( J_0 \subset I \) and \( K \subset R^{n-1} \), thus \( z \in K \). Since \( z \in Z \), the set \( I \cap \partial(E^y) \) is nonempty; let \( x' \) be any point of it. Translate \( J_0 \) a distance \( t \) (to the left or right) to a position \( J_t \subset I \) for which \( (x'; z) \in J_t \). Since \( (x'; z) \in A \), the cube \( C_t = J_t \times K \) satisfies \( \lambda_n(E \cap C_t) \geq r^{-1}\lambda_n(C_t) \). In view of \((4)\) there exists a least value of \( t > 0 \) for which the last inequality holds; denoting it by \( \eta \), we have

\[
\lambda_n(E \cap C_\eta) = r^{-1}\lambda_n(C_\eta)
\]
and
\[ \lambda_n(E \cap C_\tau) < r^{-1}\lambda_n(C_\tau) \quad \text{for } 0 < \tau < \eta. \]

By (4) and Fubini's theorem, \( \lambda_{n-1}(W) < \frac{1}{2} r^{-1}\lambda_{n-1}(K) \), where \( W = \{ y \in K : (C_0 \cap CE)^\tau = \emptyset \} \). Now consider any \( y \in K \setminus W \) such that \( (C_\eta \cap E)^\tau \neq \emptyset \). There is a point of \( \partial(E^\tau) \) between any point of \( (CE)^\tau \) and any point of \( E^\tau \), and therefore there exists \( x \in I \) such that \( x \in \partial(E^\tau) \) and \( (x; y) \in C_\tau \) where \( 0 < \tau < \eta \). Consequently \( \lambda_n(E \cap C_\tau) < r^{-1}\lambda_n(C_\tau) \), by (6), and so \( C_\tau \subset A \). This would contradict (3) if \( y \) belonged to \( Z \), so we have shown that \( \{ y : (C_\eta \cap E)^\tau \neq \emptyset \} \subset W \cup (K \setminus Z) \).

Since \( \lambda^*_{n-1}(Z \cap K) > (1 - \frac{1}{2} r^{-1})\lambda_{n-1}(K) \), it follows that
\[ \lambda^*_{n-1}(\{ y : (C_\eta \cap E)^\tau = \emptyset \}) > (1 - r^{-1})\lambda_{n-1}(K) \]
and hence by Fubini's theorem \( \lambda_n(E \cap C_\eta) < r^{-1}\lambda_n(C_\eta) \); this contradiction to (5) completes the proof.

Consider the special case where \( E \) is a totally disconnected compact set in \( R^n \) of positive \( n \)-measure. We adjust \( E \) as above to an \( x_1 \)-smooth set. In this case, the \( x_1 \)-smooth adjustment is merely a reduction of \( E \) by a set of measure zero since no linear interval intersects \( CE \) in a set of \( 1 \)-measure zero. If \( E^* \) is the \( x_1 \)-smooth reduction of \( E \), then for almost all \( y \in R^{n-1} \) the set \( \{ x : d(E, (x; y)) = 0 \text{ and } \tilde{d}(E(x; y)) = 1 \} \) is residual in \( (E^*)^\tau \) since the compactness of \( (E^*)^\tau \) implies \( (E^*)^\tau = \partial((E^*)^\tau) \). Moreover, the set of \( y \) for which \( (E^*)^\tau \) is uncountable has positive \( (n-1) \)-measure. We accordingly have the following corollary to Theorem 2.

**Corollary 1.** If \( E \) is a totally disconnected compact set of positive \( n \)-measure, then \( (\partial_x E)^\tau \) is uncountable for a set of values of \( y \) of positive \( (n - 1) \)-measure.

It is rather easy to see that the set \( \partial_x E \) of Corollary 1 is of non-\( \sigma \)-finite \( (n - 1) \)-measure. For this purpose, let \( T \) be a set in \( (n - 1) \)-space which is of finite outer Hausdorff measure, i.e. \( \lambda_0^{n-1}(T) < \infty \). Let \( N \) be a positive integer. For every \( y \in T \), let \( x_1(y) < x_2(y) < \cdots < x_N(y) \) be \( N \) reals, and let
\[ \delta(y) = \min(x_i(y) - x_{i-1}(y)), \quad i = 2, \ldots, N. \]

For every \( \delta > 0 \), let \( T_\delta \subset T \) consist of those \( y \) for which \( \delta(y) > \delta \). For each \( \eta > 0 \), there is a \( \delta > 0 \) such that \( \lambda_0^{n-1}(T_\delta) > \lambda_0^{n-1}(T) - \eta \). For each \( y \in T \), let \( S(y) = \{ (x_1(y), y), \ldots, (x_N(y), y) \} \), then \( S = \bigcup_{y \in T} S(y) \) and \( S_\delta = \bigcup_{y \in T} S(y) \).

For each covering of \( S \) by balls of radius less than \( \delta/2 \), the horizontal line at \( y \) meets \( N \) disjoint balls of the covering, for every \( y \in S_{\delta} \). This implies that \( \lambda_0^{n-1}(S) > N(\lambda_0^{n-1}(T) - \eta) \). Since this holds for every \( \eta > 0 \), we have \( \lambda_0^{n-1}(S) > N\lambda_0^{n-1}(T) \).

Suppose now that \( T \subset R^{n-1} \) is measurable and \( \lambda_0^{n-1}(T) > 0 \). For each \( y \in T \), let \( S(y) \) be uncountable and let \( S = \bigcup_{y \in T} S(y) \). Suppose \( A \subset S \), with \( \lambda_0^{n-1}(A) < \infty \). By the above, \( A \cap S(y) \) is finite for almost all \( y \in T \). Accordingly there is no sequence \( \{ A_n \} \) of subsets of \( S \) such that \( S = \bigcup_n A_n \) and \( \lambda_0^{n-1}(A_n) < \infty, n = 1, 2, \ldots \). Thus the set \( \partial_x E \) of Corollary 1 is of non-\( \sigma \)-finite \( (n - 1) \)-measure.
From Theorem 1 and Corollary 1 we now obtain

**Theorem 3.** If $E \subset \mathbb{R}^n$ is a totally disconnected compact set of positive $n$-measure then $\partial_n E$ is of non-$\sigma$-finite $(n - 1)$-measure, but there are examples of such sets for which the Hausdorff dimension of $\partial_n E$ is $n - 1$.

We benefited from very useful discussions with J. M. Marstrand at an earlier stage of this work.

**References**


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