ROWBOTTOM-TYPE PROPERTIES
AND A CARDINAL ARITHMETIC

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ABSTRACT. Assuming Rowbottom-type properties, we estimate the size of certain families of closed disjoint functions. We show that whenever \( \kappa \) is Rowbottom and \( 2^\omega < \aleph_1(\kappa) \), then \( 2^\kappa = 2^\omega \) or \( \kappa \) is the strong limit cardinal. Next we notice that every strongly inaccessible Jónsson cardinal \( \kappa \) is \( \mu \)-Rowbottom for some \( \mu < \kappa \). In turn, Shelah's method allows us to construct a Jónsson model of cardinality \( \kappa^+ \) provided \( \kappa^{cf}(\kappa) = \kappa^+ \). We include some additional remarks.

0. Introduction. In this paper the basic set-theoretical notation is standard. We only mention that the letters \( \kappa, \lambda, \mu, \ldots \) are reserved for cardinals. All undefined notions are taken from [3].

By \( j: M \rightarrow V_\gamma \), we mean an elementary embedding of the transitive model \( M \) into the collection of all sets of rank less than the limit ordinal \( \gamma \) which moves some ordinal. We write \( (\kappa, \lambda) \rightarrow (\mu, < \nu) \) iff whenever \( f: [\kappa]^\omega \rightarrow \lambda \), there exists a set \( X \subseteq \kappa \) such that \( |X| = \mu \) and \( |f''[X]|^\omega < \nu \). A cardinal \( \kappa > \nu \) is \( \nu \)-Rowbottom iff \( (\kappa, \lambda) \rightarrow (\kappa, < \nu) \) for every \( \lambda < \kappa \); \( \kappa \) is Rowbottom just in case it is \( \omega_1 \)-Rowbottom. If \( (\kappa, \nu) \rightarrow (\kappa, < \nu) \) for some \( \nu < \kappa \), then \( \kappa \) is called Jónsson.

Background information about a relation between Jónsson cardinals and elementary embeddings can be found in [8]. Mimicking [8] we can in fact prove that the property \( (\kappa, \lambda) \rightarrow (\mu, < \nu) \) is equivalent to the existence of an elementary embedding \( j: M \rightarrow V_\gamma \) such that \( j(\delta) = \lambda \) for some \( \delta < \nu \), \( j(\tau) = \kappa \) for some \( \tau \geq \mu \) and \( \mu, \nu \in j''M \) for every \( \gamma > \kappa \) (or equivalently, for some \( \gamma > \kappa \); we may additionally demand that the model \( V_\gamma \) carries countably many relations and operations).

1. Closed disjoint functions. We say that two functions \( f \) and \( g \) on \( \lambda \) are closed disjoint iff the set \( \{ \alpha < \lambda : f(\alpha) \neq g(\alpha) \} \) contains a closed unbounded subset of \( \lambda \).

**Proposition 1.1.** Assume \( j: M \rightarrow V_\gamma \), \( j(\delta) = \lambda \) for some \( \delta < \lambda \), \( \lambda \) is regular and \( \mu \in M \cap j''M \). Let \( h: \lambda \rightarrow \mu \). Then every family of closed disjoint functions \( f \in \prod_{\alpha < \lambda} h(\alpha) \) has less than \( j(\mu) \) elements.

**Proof.** Suppose the opposite: there exist a function \( h: \lambda \rightarrow \mu \) and a family of size \( j(\mu) \) of closed disjoint functions \( f \in \prod_{\alpha < \lambda} h(\alpha) \). Let \( j(\tau) = \mu \). By elementarity and absoluteness we find in \( M \) a function \( h: \delta \rightarrow \tau \) and a family \( F \in M \) of size \( \mu \) of closed disjoint-in-\( M \) functions \( f \in \prod_{\alpha < \delta} h(\alpha) \). Put \( \eta = \sup j''(\delta) < \lambda \).

We shall prove that \( (j(f))(\eta) \neq (j(g))(\eta) \) for any distinct \( f, g \in F \). Let \( C \subseteq \{ \alpha < \delta : f(\alpha) \neq g(\alpha) \} \) be closed unbounded in \( \delta \) and \( C \in M \). Therefore \( j''C \) is...
unbounded in \( \sup j''\delta = \eta \), \( j(C) \subseteq \{ \alpha < \lambda: (j(f))(\alpha) \neq (j(g))(\alpha) \} \) and \( j(C) \) is closed in \( \lambda \). Since \( j''\delta C \in j(C) \), we have \( \eta \in j(C) \).

Now we derive a contradiction, because

\[
|F| = |\{ (j(f))(\eta): f \in F \}| \leq (j(\bar{h}))(\eta) < \mu. \quad \Box
\]

In the presence of Proposition 1.1 we receive some generalizations of the known results about Chang’s Conjecture or those formulated in [7], for instance.

**Corollary 1.2.** If \( \lambda \) is a regular cardinal and \((\kappa, \lambda) \rightarrow (\mu, < \lambda)\), then every family of closed disjoint functions \( f: \lambda \rightarrow \eta \), where \( \eta < \mu \), has less than \( \kappa \) members. If in addition \( \text{cf}(\mu) > \lambda \), then every family of closed disjoint functions \( f: \lambda \rightarrow \mu \) has at most \( \kappa \) elements.

In particular, if \( \kappa \) is Rowbottom, then the above statement is true for each regular \( \omega < \lambda < \kappa = \mu \). \( \Box \)

Let \( I \) be an ideal over \( \lambda \). A partial function \( f \) on \( \lambda \) is called an \( I \)-function iff \( \text{dom}(f) \notin I \) (compare [3, p. 432]). Two \( I \)-functions \( f \) and \( g \) on \( \lambda \) are almost disjoint iff the set \( \{ \alpha < \lambda: f(\alpha) = g(\alpha) \} \) has size less than \( \lambda \).

The method of the proof of Proposition 1.1 yields

**Proposition 1.3.** Assume \( j: M \rightarrow V, j(\delta) = \lambda \) for some \( \delta < \lambda \), \( \lambda \) is regular, \( \lambda < \mu < j''\lambda \) and \( \mu \in M \). Let \( I \) be any ideal over \( \lambda \) containing all bounded subsets of \( \lambda \). Then every family of almost disjoint \( I \)-functions on \( \lambda \) into \( \mu \) has less than \( j(\mu) \) elements.

**Proof.** Argue by contradiction. Let \( j(\tau) = \mu \). By our assumption there exist an \( M \)-ideal \( I \in M \) over \( \delta \) containing all bounded subsets of \( \delta \) and a family \( F \in M \) of size \( \mu \) of almost disjoint \( I \)-functions on \( \delta \) into \( \tau \). Set \( \eta = \sup j''\delta < \lambda \).

For any distinct \( f, g \in F \) we can choose \( \beta < \delta \) such that \( f(\alpha) \neq g(\alpha) \) for all \( \alpha \in \text{dom}(f) \cap \text{dom}(g) \), \( \alpha \geq \beta \). Thus \( (j(f))(\alpha) \neq (j(g))(\alpha) \) for all \( \alpha \in \text{dom}(j(f)) \cap \text{dom}(j(g)) \), \( \alpha \geq \eta > j(\beta) \).

Since all bounded subsets of \( \lambda \) are elements of \( j(I) \), there is a subfamily \( G \subseteq F \) of size \( \mu \) and an ordinal \( \eta \leq \alpha < \lambda \) such that \( \alpha \in \text{dom}(j(f)) \) for all \( f \in G \). However, \( |G| = |\{ (j(f))(\alpha): f \in G \}| < \mu \), which is false. \( \Box \)

**Corollary 1.4.** If \( j: M \rightarrow V, j(\delta) = \lambda \) for some \( \delta < \lambda \), \( \lambda \) is regular and \( \lambda^+ \in M \), then the ideal \( I = \{ x \subseteq \lambda: |x| < \lambda \} \) of bounded subsets of \( \lambda \) is \( j(\lambda^+) \)-saturated. Hence, the ideal \( I \) is \( \lambda^+ \)-saturated, assuming \( (\lambda^+, \lambda) \rightarrow (\lambda^+, < \lambda) \).

**Proof.** If \( |x| = \lambda \), then the identity restricted to \( x \) is an \( I \)-function. Now apply Proposition 1.3. \( \Box \)

If \( f \) and \( g \) are ordinal-valued functions on a regular cardinal \( \lambda > \omega \), then the symbol \( g < f \) means that the set \( \{ \alpha < \lambda: g(\alpha) < f(\alpha) \} \) contains some closed unbounded subset of \( \lambda \) (compare [3, p. 67]). The relation \( g < f \) is well-founded and the rank \( ||f|| = \sup \{|g| + 1: g < f\} \) of \( f \) in this one is called the norm of \( f \). \( ||\tau|| \) denotes the norm of the constant function on \( \lambda \) taking \( \tau \) as the only value.

The following result can be deduced indirectly from Shelah’s work [5] using Magidor’s filters.
PROPOSITION 1.5. If \( j: M \to V_\alpha \), \( j(\delta) = \lambda \) for some \( \delta < \lambda \) and \( \lambda \) is regular, then \( \|\tau\| \leq j(\tau) \) for each \( \tau \in M \cap j"M \).

PROOF. Let \( \eta = \sup j"\delta < \lambda \). First of all we want to show that \( \|f\|_M \leq (j(f))(\eta) \) for every ordinal-valued function \( f \in M \) defined on \( \delta \). This is done by simple induction on the norm \( \|f\|_M \). If \( \|f\|_M = \beta + 1 \), then there is some function \( g \in M \) on \( \delta \) such that \( g < f \) and \( \|g\|_M = \beta \leq (j(g))(\eta) \). But the proof of Proposition 1.1 shows that \( g < f \) implies \( (j(g))(\eta) < (j(f))(\eta) \) and so \( \|f\|_M = \beta + 1 \leq (j(f))(\eta) \). The case of limit \( \|f\|_M \) is similar.

Presently, if \( \|\tau\| > j(\tau) \) and \( j(\alpha) = \tau \), then there exists a function \( g: \lambda \to \tau \) such that \( \|g\| = j(\alpha) < \tau \) and so in \( M \) there is a function \( f: \delta \to \alpha \) such that \( \|f\|_M = \tau \). But then \( \tau \leq (j(f))(\eta) < j(\alpha) = \tau \), a contradiction.

COROLLARY 1.6. If \( \lambda \) is a regular cardinal and \( (\kappa, \lambda) \to (\mu, < \lambda) \), then \( \|\mu\| \leq \kappa \). In particular, Chang’s Conjecture \( (\lambda^+, \lambda) \to (\lambda, < \lambda) \) implies \( \|\lambda\| = \lambda^+ \).

2. Cardinal exponentiation. Using some elementary embeddings we can obtain a few inequalities in cardinal arithmetic.

LEMMA 2.1. If \( j: M \to V_\gamma \), \( j(\delta) = \lambda \), \( \rho \in j"M \), \( \mu = (\rho^\delta)^+ \) and \( \mu \in M \), then \( \rho^\lambda < j(\mu) \).

PROOF. Let \( j(\eta) = \rho \) and assume to the contrary that \( \rho^\lambda \geq j(\mu) \). Hence there exists some function from \( \lambda^\rho \) onto \( j(\mu) \). So in \( M \) there is a function which transforms \( (\delta_\eta)^M \) onto \( \mu \). As \( \eta \leq \rho \), the contradiction \( \mu \leq (\delta_\eta)^M \leq \rho^\delta < \mu \) establishes the Lemma.

COROLLARY 2.2. If \( (\kappa, \lambda) \to (\mu, < \nu) \) and \( \rho^\alpha < \mu \) for all \( \alpha < \nu \), then \( \rho^\lambda < \kappa \). Therefore, if \( \kappa \) is \( \nu \)-Rowbottom and \( 2^\alpha < \kappa \) for all \( \alpha < \nu \), then \( \kappa \) is the strong limit cardinal.

REMARK 2.3. If \( 2^\omega < \aleph_\omega \) and \( \aleph_\omega \) is Rowbottom, then \( \aleph_\omega \) is the strong limit cardinal. By Theorem 84 from [3] and Corollary 1.4 we can even evaluate that \( 2^{\aleph_{\omega + 1}} = 2^{\aleph_\omega} \) or \( 2^{\aleph_{\omega + 1}} < j(\aleph_{\omega + 2}) \) for all \( n < \omega \), whenever \( j: M \to V_\gamma \) is an arbitrary elementary embedding such that \( j(\aleph_\omega) = \aleph_\omega \) and some countable ordinal is moved by \( j \).

REMARK 2.4. If a cardinal \( \kappa \) is not strong limit, then the property \( (\kappa, \lambda) \to (\kappa, < \lambda) \) fails for the least \( \lambda < \kappa \) such that \( 2^\lambda \geq \kappa \).

The same arguments can be used for proving

LEMMA 2.5. If \( j: M \to V_\gamma \), \( j(\delta) = \lambda \), \( \rho \in j"M \), \( \mu = (\delta^\rho)^+ \) and \( \mu \in M \), then \( \lambda^\rho < j(\mu) \).

COROLLARY 2.6. If \( (\kappa, \lambda) \to (\mu, < \nu) \) and \( \alpha^\rho < \mu \) for all \( \alpha < \nu \), then \( \lambda^\rho < \kappa \). Therefore, if \( \kappa \) is \( \nu \)-Rowbottom and \( \alpha^\omega < \kappa \) for all \( \alpha < \nu \), then \( \lambda^\omega < \kappa \) for all \( \lambda < \kappa \).

LEMMA 2.7. If \( (\kappa, \lambda) \to (\mu, < \lambda) \), \( \text{cf}(\nu) = \lambda \) and \( \alpha^\lambda < \mu \) for all \( \alpha < \nu \), then \( \nu^\lambda < \kappa \). Hence \( 2^\nu < \kappa \), if \( \nu \leq \mu \) is a strong limit singular cardinal with \( \text{cf}(\nu) = \lambda \).

PROOF. Choose an elementary embedding \( j: M \to V_{\kappa + \omega} \) such that \( j(\delta) = \lambda \) for some \( \delta < \lambda \), \( j(\tau) = \kappa \) for some \( \tau \geq \mu \) and \( \mu, \nu \in j"M \). But \( j(\nu) = \nu \) implies

\[
\lambda = \text{cf}(j(\nu)) = j(\text{cf}(\nu)^M(\nu)) \geq j(\lambda) > j(\delta) = \lambda.
\]
Thus \( \nu \) is moved by \( j \) and so \( j \) witnesses \((\kappa, \nu) \rightarrow (\mu, < \nu)\). Now Corollary 2.6 completes the proof. \( \square \)

Galvin-Hajnal's method \([2]\) and Corollary 1.6 allow us to formulate some bounds on \( \nu^{\text{cf}(\nu)} \) in certain cases of singular cardinals \( \nu > \mu \).

**Theorem 2.8.** Assume \((\kappa, \lambda) \rightarrow (\mu, < \lambda)\). Let \( \lambda \leq \eta \leq \mu \) and \( \nu = \aleph_\eta \). If \( \text{cf}(\eta) = \lambda \) and \( \alpha^\lambda < \nu \) for all \( \alpha < \nu \), then \( \nu^\lambda < \aleph_\kappa \). In particular, if \( \nu \) is the strong limit singular cardinal of cofinality \( \lambda \), then \( 2^\nu < \aleph_\kappa \). \( \square \)

**Corollary 2.9 (Magidor [4]).** Presuming Chang's Conjecture \((\lambda^+, \lambda) \rightarrow (\lambda, < \lambda), \text{ if } \aleph_\lambda \text{ is the strong limit cardinal, then } 2^{\aleph_\lambda} < \aleph_{\lambda^+}. \square \)

**Corollary 2.10.** If \( \aleph_\omega \) is Rowbottom and \( \aleph_{\omega_n} \) is the strong limit cardinal for some \( n < \omega \), then \( 2^{\aleph_{\omega_n}} < \aleph_{\omega_\omega}. \square \)

The main theorem is based on the following technical

**Claim 2.11.** Let \( \lambda \) be the least cardinal such that \( \rho^\lambda > \rho^\nu \). If \((\kappa, \lambda) \rightarrow (\mu, < \lambda), \lambda < \mu, \text{ cf}(\mu) \neq \lambda \) and \( \nu^\mu < \aleph_\lambda \), then \( \nu^\lambda < \kappa \). Moreover, \( \rho^\lambda < \kappa \) unless \( \kappa \) is singular.

**Proof.** Simple arithmetic shows that \( \lambda \) is regular. Clearly, \( \nu < \lambda \leq \rho^\nu \). Assigning for each \( f \in \lambda^\rho \) the sequence \( \bar{f} = \{ f \upharpoonright \alpha: \alpha < \lambda \} \) we get the branching family \( F \) of \( \rho^\lambda \) functions from \( \lambda \) into some set of cardinality \( \rho^\nu \) (whenever \( \bar{f}, \bar{g} \in F \) and \( \bar{f}(\beta) = \bar{g}(\beta) \), then \( \bar{f}(\alpha) = \bar{g}(\alpha) \) for all \( \alpha < \beta \)—compare [3, p. 431]).

Let \( \rho^\nu < \sigma < \rho^\lambda \) be any regular cardinal. As \( \text{cf}(\eta) \neq \lambda \) for all cardinals \( \mu \leq \eta < \aleph_\lambda(\mu) \), the proof of Lemma 35.2 in [3] shows how then to construct a branching family \( G \) of \( \sigma \) functions \( f: \lambda \rightarrow \eta \) for some \( \eta \leq \mu, \eta \leq \lambda \) or \( \text{cf}(\eta) = \lambda \). But Corollary 1.2 gives \(|G| < \kappa \), so we are done. \( \square \)

**Lemma 2.12.** If \( \nu < \mu, \text{ cf}(\mu) \leq \nu \) or \( \mu \) is regular, \((\kappa, \lambda) \rightarrow (\mu, < \lambda)\) holds for every regular \( \nu < \lambda < \mu \) and \( \kappa^- \leq \rho^\nu < \aleph_{\nu^+}(\mu) \), then \( \rho^{<\mu} = \rho^\nu \). \( \square \)

**Corollary 2.13.** Assume \((\lambda^+, \lambda) \rightarrow (\lambda^+, < \lambda)\). Then \( 2^{<\lambda} = \lambda \) implies \( 2^\lambda = \lambda^+ \) and \( \lambda^+ \leq 2^{<\lambda} < \aleph_\lambda \) implies \( 2^\lambda = 2^{<\lambda}. \square \)

**Theorem 2.14.** Let \( \nu = \sup(\lambda < \kappa: \lambda \text{ is regular and } (\kappa, \lambda) \rightarrow (\kappa, < \lambda) \text{ fails}). \) If \( \nu < \kappa \) and \( \kappa^- \leq 2^\nu < \aleph_{\nu^+}(\kappa) \), then \( 2^{<\kappa} = 2^\nu \). In particular, if \( \kappa \) is \( \nu^+ \)-Rowbottom and \( \kappa \leq 2^\nu < \aleph_{\nu^+}(\kappa) \), then \( 2^{<\kappa} = 2^\nu \).

**Proof.** If \( \nu < \kappa \) then \((\kappa, \lambda) \rightarrow (\kappa, < \lambda)\) for every regular \( \nu < \lambda < \kappa \). It is easy to see that then \( \kappa \) is regular or \( \text{cf}(\kappa) \leq \nu \) (compare Lemma 2 in [8]). Applying Lemma 2.12 we finish the proof. \( \square \)

**Corollary 2.15.** If \( \aleph_\omega \) is Rowbottom and \( \aleph_\omega < 2^{\aleph_n} < \aleph_{\omega_{n+1}} \) for some \( n < \omega \), then \( 2^{\aleph_n} = 2^{\aleph_n} \). \( \square \)

**Remark 2.16.** If \( \kappa \) is \( \nu \)-Rowbottom and \( \kappa \leq 2^{<\nu} < \aleph_{\text{cf}(\nu)}(\kappa) \), then \( 2^{<\kappa} = 2^{<\nu}. \) \( \square \)

**Question 2.17.** Under the notation of Theorem 2.14, is \( \nu < \kappa \) whenever \( \kappa \) is Jónsson?
3. The first critical point. The least ordinal moved by an elementary embedding \( j: M \rightarrow V_\gamma \) is regular in \( M \). Thus the correspondence between Rowbottom-type properties and elementary embeddings shows that the first cardinal \( \nu \leq \mu \) such that \( (\kappa, \nu) \rightarrow (\mu, < \nu) \) is regular. We shall prove that such \( \nu \) cannot be strongly inaccessible, whenever \( \mu = \kappa \).

**Lemma 3.1.** Assume that \( (\kappa, \nu) \rightarrow (\kappa, < \nu) \) and \( \nu < \kappa \) is a limit cardinal. Let \( \lambda \leq \kappa \) be the least cardinal such that \( \lambda > \nu \) and the property \( (\kappa, \lambda) \rightarrow (\kappa, < \lambda) \) fails. If \( \nu^{\text{cf}}(\nu) < \lambda \), then there exists \( \rho < \nu \) such that \( (\kappa, \nu) \rightarrow (\kappa, < \rho) \).

**Proof.** Set \( A = \{ \sigma < \nu: (\kappa, \sigma) \rightarrow (\kappa, < \sigma) \} \) and \( \eta = \sup A \). If \( \eta < \nu \), then our statement is true for \( \rho = \eta^+ \). If this were false, there would be some strictly increasing sequence \( \langle \sigma_\alpha: \alpha < \text{cf}(\nu) \rangle \) of elements of \( A \), cofinal in \( \nu \). For each \( \alpha < \text{cf}(\nu) \) we can find a partition \( f_\alpha: [\kappa]^\kappa \rightarrow \sigma_\alpha \) such that \( |f_\alpha''[X]^{<\omega}| = \sigma_\alpha \) for every \( X \subseteq \kappa \) of size \( \kappa \) (a counterexample for \( (\kappa, \sigma_\alpha) \rightarrow (\kappa, < \sigma_\alpha) \)).

Put \( B = \prod_{\alpha < \text{cf}(\nu)} \sigma_\alpha \) and define \( g: [\kappa]^\kappa \rightarrow B \) setting \( g(s) = \langle f_\alpha(s): \alpha < \text{cf}(\nu) \rangle \) for each finite subset \( s \subseteq \kappa \). As \( |B| = \nu^{\text{cf}}(\nu) < \lambda \), the definition of \( \lambda \) supplies a set \( X \subseteq \kappa \) such that \( |X| = \kappa \) and \( |g''[X]^{<\omega}| = \nu \). On the other hand, \( |g''[X]^{<\omega}| \geq \sup_{\alpha < \text{cf}(\nu)} |f_\alpha''[X]^{<\omega}| \geq \nu \) by definition of \( g \). This contradiction establishes the Lemma. \( \square \)

**Corollary 3.2.** If \( \nu < \kappa \) is a strong limit cardinal and \( (\kappa, \nu) \rightarrow (\kappa, < \nu) \), then \( (\kappa, \nu) \rightarrow (\kappa, < \rho) \) for some \( \rho < \nu \). Thus the least cardinal \( \lambda < \kappa \) such that \( (\kappa, \lambda) \rightarrow (\kappa, < \lambda) \) cannot be strongly inaccessible.

**Proof.** Let \( \lambda \leq \kappa \) be the least cardinal such that \( \lambda > \nu \) and \( (\kappa, \lambda) \rightarrow (\kappa, < \lambda) \) fails. Observe that \( \lambda \) is \( \nu \)-Rowbottom. Since \( 2^{\nu^+} = \nu \), the cardinal \( \lambda \) is strong limit by Corollary 2.2. Now \( \nu^{\text{cf}}(\nu) = 2^\nu < \lambda \) and our claim follows from Lemma 3.1. \( \square \)

**Question 3.3.** Is the least \( \lambda < \kappa \) such that \( (\kappa, \lambda) \rightarrow (\kappa, < \lambda) \) is always a successor cardinal?

**Theorem 3.4.** Every strongly inaccessible Jónsson cardinal \( \kappa \) is \( \rho \)-Rowbottom for some \( \rho < \kappa \).

**Proof.** Let \( \lambda < \kappa \) be any regular cardinal such that \( (\kappa, \lambda) \rightarrow (\kappa, < \lambda) \). The set \( S = \{ \nu < \kappa: \text{cf}(\nu) = \lambda \) and \( \nu \) is strong limit \} is stationary in \( \kappa \). Lemma 2 from \([8]\) shows that \( (\kappa, \nu) \rightarrow (\kappa, < \nu) \) for every \( \nu \in S \). By Corollary 3.2, for each \( \nu \in S \) we can choose some \( \rho_\nu < \nu \) so that \( (\kappa, \nu) \rightarrow (\kappa, < \rho_\nu) \). By Fodor’s Theorem there exist some fixed \( \rho < \kappa \) and a stationary subset \( T \subseteq \kappa \) such that \( (\kappa, \nu) \rightarrow (\kappa, < \rho) \) for each \( \nu \in T \). This means that \( \kappa \) is \( \rho \)-Rowbottom, since \( T \) is unbounded in \( \kappa \). \( \square \)

**Question 3.5.** May we erase the word “strongly” from the above Theorem?

4. Jónsson models and successor cardinals. We showed in \([8]\) that a successor cardinal \( \kappa^+ \) is not Jónsson, whenever \( \kappa \) is regular. Alternatively, \( \kappa^+ \) is not Jónsson under \( 2^\kappa = \kappa^+ \) \([1]\). Shelah’s method from \([6]\) enables us to weaken this presumption.

We say that a regular cardinal \( \mu \) is a possible scale for the sequence \( \langle \kappa_i: i < \lambda \rangle \) of cardinals iff there exists a sequence \( \langle f_\alpha: \alpha < \mu \rangle \) of functions on \( \lambda \) such that

\[(i) \ f_\alpha \in \prod_{i < \lambda} \kappa_i \text{ for all } \alpha < \mu, \]
\[(ii) \ |\{i < \lambda: f_\alpha(i) \geq f_\beta(i)\}| < \lambda \text{ for all } \alpha < \beta < \mu. \text{ (We shall write } f_\alpha < f_\beta.\text{)} \]
\[(iii) \text{ For every } f \in \prod_{i < \lambda} \kappa_i \text{ there exists } \alpha < \mu \text{ such that } |\{i < \lambda: f(i) \leq f_\alpha(i)\}| = \lambda \text{ (compare [6]).} \]
LEMMA 4.1 (SHELAH [6]). Let $j: M \to V$, and $N = j''M$. If $\mu$ is a possible scale for the sequence $(\kappa_i: i < \lambda) \in N$ of regular cardinals, $\lambda + 1 \subseteq N$ and $j(\mu) = \mu$, then $\{i < \lambda: j(\kappa_i) = \kappa_i\} = \lambda$.

PROOF. Set $A = \{i < \lambda: |N \cap \kappa_i| < \kappa_i\}$ and assume to the contrary that $|A| < \lambda$. As $N$ is the elementary substructure of $V$, some sequence $(f_\alpha: \alpha < \mu) \in N$ exemplifies that $\mu$ is a possible scale for $(\kappa_i: i < \lambda)$. For each $i \in A$ the subset $B_i = \{f_\alpha(i): \alpha \in N \cap \mu\}$ of $N \cap \kappa_i$ has cardinality less than $\kappa_i$, so we may choose $\sup B_i < f(i) < \kappa_i$ by regularity of $\kappa_i$. Accepting $f(i) = 0$ for $i \in \lambda \setminus A$, we have $f_\alpha < f$ for every $\alpha \in N \cap \mu$. As the relation $<$ is transitive and the set $N \cap \mu$ is cofinal in $\mu$, $f_\alpha < f$ for every $\alpha < \mu$, which is impossible. □

THEOREM 4.2. If $\kappa^{cf(\kappa)} = \kappa^+$, then $\kappa^+$ is not Jónsson.

PROOF. By our first remark in this item we may assume that $\lambda = cf(\kappa) < \kappa$. Suppose that $\kappa^+$ is Jónsson and pick an elementary embedding $j: M \to V_{\kappa^+ + \omega}$ such that $j(\alpha) = \alpha$ for all $\alpha \leq \lambda$, $j(\nu) > \nu$ for some $\nu < \kappa$ and $j(\kappa^+) = \kappa^+$ (see the proof of Theorem 1 in [8]). Then there exists some strictly increasing sequence $(\kappa_i: i < \lambda) \in j''M$ of cardinals, cofinal in $\kappa$, with $\kappa_0 \geq \nu$.

Cantor's diagonalization method shows that every family of $\kappa$ functions $f \in \prod_{i < \lambda} \kappa_i^{+\omega}$ has an upper bound in the relation $<$. Since $|\prod_{i < \lambda} \kappa_i^{+\omega}| = \kappa^\lambda = \kappa^+$, $\kappa^+$ is the only possible scale for $(\kappa_i^+: i < \lambda)$. Now, by Lemma 4.1, the set $A = \{i < \lambda: j(\kappa_i^+) = \kappa_i^+\}$ is unbounded in $\lambda$. But each $\kappa_i^+$, where $i \in A$, is Jónsson, contradicting [8]. □

COROLLARY 4.3 (SHELAH [6]). If $2^{\aleph_\alpha} = \aleph_{\alpha + \gamma + 1}$ and $cf(\gamma) < \aleph_{\alpha + 1}$, then $2^{\aleph_\alpha}$ cannot be Jónsson. □

With a slight modification, a similar argument can be used for

LEMMA 4.4. If $\kappa^+$ is Jónsson, $\lambda = cf(\kappa) > \omega$ and $\rho^\lambda < \kappa$ for $\rho < \kappa$, then the set $\{\rho < \kappa: \rho^+ is Jónsson\}$ contains some closed unbounded subset of $\kappa$.

PROOF. Let $j: M \to V_{\kappa^+ + \omega}$, $j(\alpha) = \alpha$ for all $\alpha \leq \lambda$, $j(\nu) > \nu$ for some $\lambda < \nu < \kappa$ and $j(\kappa^+) = \kappa^+$. Choose a strictly increasing continuous sequence $(\rho_i: i < \lambda) \in j''M$ of cardinals, cofinal in $\kappa$, with $\rho_0 \geq \nu$. Suppose by way of contradiction that the set $S = \{i < \lambda: \rho_i^+ is not Jónsson\}$ is stationary in $\lambda$.

Set $\kappa_i = \rho_i^+$ for $i \in S$ and $\kappa_i = \rho_i^{i+}$ for $i \in \lambda \setminus S$. Thus no $\kappa_i$ is Jónsson. Since $S$ is stationary, by an analogue of Lemma 8.5 stated in [3], every family of almost disjoint functions $f \in \prod_{i < \lambda} \kappa_i$ has at most $\kappa^+$ elements. But every family of $\kappa$ functions $f \in \prod_{i < \lambda} \kappa_i$ has an upper bound in the relation $<$. Thus $\kappa^+$ is the only possible scale for $(\kappa_i: i < \lambda)$. Now each element of the set $\{\kappa_i: i < \lambda \land j(\kappa_i) = \kappa_i\} \neq \emptyset$ is Jónsson, contrary to our choice. □

REMARK 4.5. We recall another result of Shelah from [6] which can be formulated as follows: If $\rho^{cf(\kappa)} < \kappa$ for all $\rho < \kappa$ and $\kappa^+$ is Jónsson, then the set $\{\lambda < \kappa: \lambda is a regular Jónsson cardinal\}$ is unbounded in $\kappa$. Hence, if $\lambda$ is arbitrary and $\lambda^{\omega} = \aleph_\alpha$, then $\aleph_{\alpha + \omega + 1}$ cannot be Jónsson. □

We can also leave out one assumption in Shelah's result [6].

LEMMA 4.6. If $(\lambda^+)^\omega = \lambda^+$ for all singular cardinals $\lambda$, then no successor cardinal is Jónsson.
PROOF. Suppose to the contrary that $\kappa^+$ is the least successor cardinal which is Jónsson. There are now two cases:

Case I: $\cf(\kappa) = \omega$. Then $\kappa^{\cf(\kappa)} \leq (\kappa^+)^\omega = \kappa^+$ and a contradiction follows from Theorem 4.2.

Case II: $\kappa > \cf(\kappa) > \omega$. First, collapse $\cf(\kappa)$ onto $\omega_1$ using the notion of forcing $P = \{ p : p \text{ is a function with } \dom(p) \subseteq \omega_1 \text{ and } \ran(p) \subseteq \cf(\kappa) \}$ ordered by inclusion.

Let $G$ be any generic filter on $P$. Since $|P| = \cf(\kappa)^\omega < \kappa$, it follows from [8] that $\kappa^+$ remains Jónsson in the forcing extension $V[G]$. Clearly, since $P$ is $\omega_1$-closed, $\omega_1$ is preserved and $\cf(\kappa) = \omega_1$ in $V[G]$. Moreover, $|P|$ is collapsed onto $\omega_1$ and every cardinal $\lambda > |P|$ in $V$ remains a cardinal in $V[G]$. The equality $(\lambda^+)^\omega = \lambda^+$ is also true in $V[G]$ for every singular cardinal $\lambda$.

From now on work in $V[G]$. Let $j : M \to V, \gamma$ witness that $\kappa^+$ is Jónsson. Pick some strictly increasing continuous sequence $(\kappa_i : i < \omega_1) \in j''M$ of cardinals with cofinality $\omega$, cofinal in $\kappa$. Since $\prod_{i < 1} \kappa_i^+ \leq (\kappa_1^+)^\omega = \kappa_1^+$ for all $1 < \omega_1$, by Claim 13 from [6] the successor $\kappa_1^+$ is a possible scale for $(\kappa_i^+ : i < \omega_1)$. It follows from Lemma 4.1 that the set $A = \{ i < \omega_1 : j(\kappa_i^+) = \kappa_i^+ \}$ is unbounded in $\omega_1$. But if $i \in A$ and $\kappa_i^+$ is greater than the first ordinal moved by $j$, then $\kappa_i^+$ is Jónsson, which is a contradiction because $\cf(\kappa_i) = \omega$ and $(\kappa_i)^\omega = \kappa_i^+$. \(\square\)

The same proof shows

**Lemma 4.7.** Assume the Singular Cardinals Hypothesis. Then $2^\omega < \kappa$ implies that $\kappa^+$ is not Jónsson. \(\square\)

**Lemma 4.8.** No successor cardinal above a compact cardinal is Jónsson.

**Proof.** Solovay showed that the singular cardinals hypothesis holds above the least compact cardinal (see [3, p. 405]). Now proceed as in the proof of Lemma 4.6. \(\square\)

**References**


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