MAPS WHICH PRESERVE ANR'S

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Abstract. It is shown that maps preserve ANR's and LC−property if they satisfy a certain movability condition in the fiber shape theory. This generalizes the known results of hereditary shape equivalences to a non cell-like case.

1. Introduction. Spaces are assumed to be metrizable and ANR's are ones for metric spaces. Suppose \( f: X \to Y \) is a proper onto map (\( f^{-1}(B) \) is compact for each compact \( B \subset Y \)) and \( X \) is an ANR. It is a long-standing problem to determine conditions on \( f \) under which \( Y \) is an ANR. G. Kozlowski [K] proved that if \( f \) is a hereditary shape equivalence then \( Y \) is an ANR. In [Y1] we introduced the notion of movability for maps and proved [Y1, Theorem 1.2] that \( f \) is a hereditary shape equivalence iff \( f \) is a CE-map and movable. Here an onto map \( f: X \to Y \) is said to be movable provided for some (eq. any) ANR \( M \) containing \( X \) as a closed subset the following holds:

\[ (*) \text{ For each neighborhood } U \text{ of } f^{-1} = \bigcup \{ \{ y \} \times f^{-1}(y) : y \in Y \} \text{ in } Y \times M \text{ there exists a neighborhood } V \text{ of } f^{-1} \text{ in } U \text{ such that for each neighborhood } W \text{ of } f^{-1} \text{ in } V \text{ there exists a homotopy } h: V \times [0,1] \to U \text{ such that } h_0 = \text{id}, h_1(V) \subset W, ph_t = p \quad (0 < t < 1), \text{ where } p: Y \times M \to Y \text{ is the projection.} \]

In addition, if we can take the homotopy \( h \) so that \( h_1|_{f^{-1}} = \text{id} \quad (0 \leq t \leq 1) \), then we say \( f \) is strongly movable.

The purpose of this note is to show that this movability assumption on \( f \) is sufficient to ensure that \( Y \) is an ANR.

Theorem. Suppose \( f: X \to Y \) is a movable map.

(1) If \( X \) is an ANR then so is \( Y \).

(2) If \( X \) is locally \( n \)-connected (LC\(^n\)) then so is \( Y \).

In [K], it was also proved that a CE-map with an ANR domain is a hereditary shape equivalence iff the range is an ANR. Our theorem, combined with [CD1, Proposition 3.6], [Y1, Corollary 4.4], yields the following version.

Corollary. Suppose \( f: X \to Y \) is a proper onto map and \( X \) is a separable, locally compact ANR. Then \( f \) is (strongly) movable iff \( Y \) is an ANR and \( f \) is completely movable.
As for the definition of the complete movability, refer to [CD], and for the other related topics, refer to [Y].

Remark. (i) In the above corollary, the complete movability of $f$ implies the approximate homotopy lifting property for all $n$-cells ($n > 0$) [CD, Theorem 3.3, Proposition 3.6] and also that each fiber of $f$ is an FANR. Therefore $Y$ is $LC^\infty$ by [CD, Theorem 3.4]. However, in general, $Y$ is not necessarily an ANR, because there exists a CE (hence completely movable) map from the Hilbert cube to a compactum which is not a shape equivalence [T].

(ii) In [Y2], it is shown that any movable map does not raise dimension. Therefore, if $f: X \to Y$ is a completely movable map and $X$ is an $n$-dimensional, locally compact ANR, then $Y$ is an ANR if and only if $\dim Y \leq n$ [Y2, Corollary 3.5].

2. Proof of Theorem. Suppose $f: X \to Y$ is a movable map and $M$ and $N$ are ANR's which contain $X$ and $Y$ as closed subsets respectively. Let $p: N \times M \to N$ denote the projection and let $p$ be a metric on $N$. For each neighborhood $U$ of $f^{-1}$ in $N \times M$, we define $U|_Y = U \cap Y \times M$. First we have a lemma.

Lemma 1. Let $W_i$ ($i > 0$) be open neighborhoods of $f^{-1}$ in $N \times M$. Then there exist an open neighborhood $U$ of $f^{-1}$ in $Y \times M$, open neighborhoods $V_i$ ($i > 0$) of $f^{-1}$ in $N \times M$ and a map $h$: $W_0 \times [0, \infty) \to W_0$ such that $V_{i+1} \subset V_i$, $U = V_1|_Y$, $h_0 = \text{id}$, $ph_i = p$, $h(V_i \times [i, \infty)) \subset W_i$ ($i > 0$).

Proof. We may assume $W_{i+1} \subset W_i$ ($i > 0$). Since $f$ is movable, there exist open neighborhoods $U_i$ ($i > 0$) of $f^{-1}$ in $Y \times M$ and homotopies $g_i$: $U_i \times [0, 1] \to U_{i-1}$ ($i > 1$) such that $U_{i+1} \subset U_i \subset W_i|_Y$ ($i > 0$), $g_0 = \text{id}$, $g_i|_U \subset U_{i+1}$, $pg_i = p$ ($i > 1$, $0 < t < 1$). Let $U = U_1$ and define $g$: $U \times [0, \infty) \to W_0$ by

$$g(y, x, t) = g^n(g^{n-1} \circ \cdots \circ g_1(y, x), t - (n - 1)) \in U_{n-1}$$

for $(y, x) \in U$, $n - 1 \leq t \leq n$, $n > 1$.

Take an open neighborhood $W_0$ of $f^{-1}$ in $W_0$ with $W_0|_Y = U$. Since $M$ is an ANR, using the Borsuk homotopy extension theorem or its proof [H, p. 117], inductively we can find maps $h^n$: $V_0 \times [0, n] \to W_0$ ($n > 1$) and open neighborhoods $V_n$ ($n > 1$) of $U$ in $V_0$ such that $h^{n+1}|_{V_0 \times [0, n]} = h^n$, $h_0^n = \text{id}$, $h^n|_{U \times [0, n]} = g|_{U \times [0, n]}$, $ph_i^n = p$ ($0 \leq s \leq n$), $h^n(V_i \times [i, n]) \subset W_i$ ($0 \leq i \leq n$). The desired map $h$ is obtained by piecing $h^n$ ($n > 1$) together.

Proof of Theorem (1). Since the ANR $f^{-1} \times X$ is closed in $N \times M$, there exists a retraction $r$: $W_0 \to f^{-1}$ from some open neighborhood of $f^{-1}$ in $N \times M$. Since $pr|_{f^{-1}} = p|_{f^{-1}}$, we can find open neighborhoods $W_i$ ($i > 1$) of $f^{-1}$ in $W_0$ such that $r(pr(z), p(z)) < 1/i$ ($z \in W_i$, $i > 1$). Applying Lemma 1 to $W_i$ ($i > 0$), we obtain $U, V_i$ ($i > 0$) and $h$ as in Lemma 1.

Let $y_0 \in Y$. Take a point $x_0 \in f^{-1}(y_0)$ and an open neighborhood $K$ of $y_0$ in $N$ such that $K \times \{x_0\} \subset V_0$. We will show that there exists a map $k$: $K \to Y$ such that $k|_{K \cap Y} = \text{id}_{K \cap Y}$. Then $K \cap Y$ is an ANR neighborhood of $y_0$ in $Y$ since $k|_{K \cap Y}$ is a retraction from an ANR $k^{-1}(K \cap Y)$ onto $K \cap Y$. This implies that $Y$ is a local ANR, hence an ANR [H, p. 68].
Define a map \( s: K \to V_0 \) by \( s(y) = (y, x_0) \) (\( y \in K \)). Since \( s(K \cap Y) \subseteq U \subseteq V_i \) (\( i \geq 0 \)), there exist closed neighborhoods \( K_i \) (\( i \geq 0 \)) of \( K \cap Y \) in \( K \) such that \( K_0 = K, \ K_{i+1} \subseteq \text{Int} \ K_i, \ K_i \cap K_{i+1} = K \cap Y \) and \( s(K_i) \subseteq V_i \) (\( i \geq 1 \)). Take a function \( \lambda: K \to [0, \infty) \) with \( \lambda(K_i - Y) \subseteq [i, \infty) \) and define \( k: K \to Y \) by

\[
k(y) = \begin{cases} y, & y \in K \cap Y, \\ \text{prh}(s(y), \lambda(y)), & y \in K - Y. \end{cases}
\]

If \( y \in K_i - Y, \ i \geq 1 \), then \( h(s(y), \lambda(y)) \in W_i \) and by the choice of \( W_i \), \( \rho(k(y), y) < 1/i \). The continuity of \( k \) follows from this observation. This completes the proof.

We proceed to the proof of Theorem (2) and assume \( f^{-1} \equiv X \) is \( LC^n \). If \( \mathcal{U} \) is an open cover of \( f^{-1} \) in \( N \times M \), then two maps \( g, g': P \to f^{-1} \) are said to be \( \mathcal{U}\)-near if for each \( x \in P \) there exists \( U \in \mathcal{U} \) such that \( g(x), g'(x) \in U \). The next lemma follows from [H, p. 156, Theorem 4.1] and will play the same role as the retraction \( r: W_0 \to f^{-1} \) in the preceding proof.

**Lemma 2.** Let \( \mathcal{U}_i \) (\( i \geq 0 \)) be a sequence of open coverings of \( f^{-1} \) in \( N \times M \). Then there exist open neighborhoods \( W_i \) (\( i \geq 0 \)) of \( f^{-1} \) in \( N \times M \) such that if \( P = \bigcup \{ P_i: i \geq 0 \} \) is an \((n+1)\)-dimensional locally compact polyhedron, \( P_i \) is a compact subpolyhedron of \( P \), \( P_i \subseteq \text{Int} P_{i+1} \) (\( i \geq 0 \)) and \( g: P \to W_0 \) is a map with \( g(P_i - \text{Int} P_{i-1}) \subseteq W_i \) (\( i \geq 0 \)), then there exists a map \( g': P \to f^{-1} \) such that \( g \) and \( g' \) are \( \mathcal{U}_i\)-near on \( P_i - \text{Int} P_{i-1} \) for \( i \geq 0 \).

**Proof of Theorem (2).** To see \( Y \) is \( LC^n \), let \( y_0 \in Y \) and let \( L_0 \) be any neighborhood of \( y_0 \) in \( Y \). Take open neighborhoods \( K_0, K_1 \) of \( y_0 \) in \( N \) such that \( K_0 \cap Y = L_0, \text{Cl} K_1 \subseteq K_0 \) (\( \text{Cl} K_1 \) is the closure of \( K_1 \) in \( N \)).

For each \( i \geq 1 \) take an open covering \( \mathcal{U}_i \) of \( N \times M \) which refines \( \mathcal{U}_0 = \{(N - \text{Cl} K_1) \times M, \ K_0 \times M \} \) and such that for each \( U \in \mathcal{U}_i \), \( \text{diam} p(U) < 1/i \). There exist open neighborhoods \( W_i \) (\( i \geq 0 \)) of \( f^{-1} \) in \( N \times M \) as in Lemma 2. Then there exist \( U, V_i \) (\( i \geq 0 \)) and \( h \) as in Lemma 1. Take a point \( x_0 \in f^{-1}(y_0) \) and open neighborhoods \( K_2 \subseteq K_1 \) of \( y_0 \) in \( K_1 \) such that \( K_2 \times \{x_0\} \subseteq V_0 \) and the inclusion \( K_3 \subseteq K_2 \) is nullhomotopic (note that the ANR \( N \) is locally contractible).

Let \( L = K_3 \cap Y \). We will show that any map \( \alpha: S^k \to L \) from the \( k \)-sphere \( S^k \) (\( 0 \leq k \leq n \)) has an extension \( \beta: B^{k+1} \to L_0 \) over the \((k+1)\)-ball \( B^{k+1} \).

Since \( \alpha \) is nullhomotopic in \( K_2 \), \( \alpha \) extends to a map \( \gamma: B^{k+1} \to K_2 \). Define \( s: B^{k+1} \to V_0 \) by \( s(z) = (\gamma(z), x_0), \ z \in B^{k+1} \). Since \( s(S^k) \subseteq U \subseteq V_i \) (\( i \geq 0 \)), there exist compact subpolyhedra \( P_i \) (\( i \geq 0 \)) of \( B^{k+1} \) (the interior of \( B^{k+1} \)) such that \( B^{k+1} = \bigcup P_i, \ P_i \subseteq \text{Int} P_{i+1}, \ s(\text{Int} P_i) \subseteq V_{i+1} \) (\( i \geq 0 \)). Take a function \( \lambda: B^{k+1} \to [0, \infty) \) such that \( \lambda(\text{Int} P_i) \subseteq [i+1, \infty) \). Define \( g: B^{k+1} \to W_0 \) by \( g(z) = h(s(z), \lambda(z)) \). Then \( g(P_i - \text{Int} P_{i-1}) \subseteq W_i \) (\( i \geq 0 \)). By the choice of \( W_i \) (\( i \geq 0 \)), we have a map \( g': B^{k+1} \to f^{-1} \) such that \( g \) and \( g' \) are \( \mathcal{U}_i\)-near on \( P_i - \text{Int} P_{i-1} \) (\( i \geq 0 \)). Note that \( pg = \gamma, \ pg'(\hat{B}^{k+1}) \subseteq L_0, \rho(pg'(z), \gamma(z)) < 1/i \) (\( z \in P_i - \text{Int} P_{i-1} \), \( i \geq 1 \)). Finally define the map \( \beta: B^{k+1} \to L_0 \) by \( \beta|_{\hat{B}^{k+1}} = \alpha, \beta|_{B^{k+1}} = pg' \). The continuity of \( \beta \) follows from the above observation. This completes the proof.
REFERENCES


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