ON CERTAIN GROUPS OF CENTRAL TYPE
ALBERTO ESPUELAS

ABSTRACT. A finite group $G$ is a group of central type if there exists $\chi \in \text{Irr}(G)$ with $\chi(1)^2 = |G: Z(G)|$. It is known that, in such conditions, $G$ is solvable. Here some conditions assuring the nilpotence of groups of central type are given.

Let $G$ be a finite group, $N$ a normal subgroup of $G$ and $\theta \in \text{Irr}(N)$ invariant in $G$. Then we say that $(G, N, \theta)$ is a character triple. Such a triple is fully ramified provided that one of the following (equivalent) conditions holds:

(i) If $\chi \in \text{Irr}(G|\theta)$ then $\chi$ vanishes off $N$.
(ii) $\theta^G$ has a unique irreducible constituent $\chi$.
(iii) If $\chi \in \text{Irr}(G|\theta)$ then $(\chi(1)/\theta(1))^2 = |G: N|$ (see [5, p. 95]).

We say that $\chi$ is fully ramified in $N$ and $\theta$ is fully ramified in $G$. A group $G$ is of central type provided that there exists $\chi \in \text{Irr}(G)$ with $\chi(1)^2 = |G: Z(G)|$. If $\lambda$ is the irreducible constituent of $\chi_Z(G)$ then $(G, Z(G), \lambda)$ is fully ramified. It is well known (see Lemma 4.3 of [3]) that if $(G, N, \theta)$ is fully ramified then there exists a group $G_1$ of central type and $\lambda \in \text{Irr}(Z(G_1))$ such that $(G, N, \theta)$ and $(G_1, Z(G_1), \lambda)$ are isomorphic character triples (see [5, p. 187] for the definition).

In [3] Howlett and Isaacs introduced the concept of an irreducible group of central type. We recall that definition: Let $G$ be a group of central type with $\lambda \in \text{Irr}(Z(G))$ fully ramified in $G$. Then $G$ is reducible if $\lambda$ is fully ramified in some normal subgroup $N$ of $G$ with $Z(G) < N < G$. Otherwise, $G$ is irreducible.

In [2] Gagola raised the question about the nilpotence of irreducible groups of central type. The answer is negative as Example 8.2 of [3] shows. Our result is the following

THEOREM. Let $G$ be an irreducible group of central type, $Z = Z(G)$ and $\overline{G} = G/Z$. Let $\lambda \in \text{Irr}(Z(G))$ be fully ramified in $G$. Then

(i) If there exists a prime $p$ such that $|\text{socle}(\overline{G})|_p > |\overline{G}: \text{socle}(\overline{G})|_p$ then $G$ is nilpotent.
(ii) $F(\overline{G}/\text{socle}(\overline{G})) = F(\overline{G})/\text{socle}(\overline{G})$, where $F(X)$ denotes the Fitting subgroup of $X$. As a consequence, if $\overline{G}/\text{socle}(\overline{G})$ has a (nontrivial) normal Sylow subgroup then $G$ is nilpotent.

We need the following

LEMMA. Let $G$ be an irreducible group of central type, $Z = Z(G)$ and $\overline{G} = G/Z$. Let $\lambda \in \text{Irr}(Z(G))$ be fully ramified in $G$. Then, assuming that $G$ is not nilpotent, we have that $\lambda$ is extendible to the preimage of $\text{socle}(\overline{G})$ in $G$. 

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PROOF OF THE LEMMA. The group $G$ is solvable by Theorem 7.3 of [3]. We may assume that $\lambda$ is faithful.

First we show that $\lambda$ is extendible to the preimage $A$ in $G$ of any minimal normal subgroup $\overline{A}$ of $\overline{G}$. As $\overline{G}$ is solvable, $\overline{A}$ is an elementary abelian group. Then there exists a subgroup $A_1$ normal in $G$ with $Z \leq A_1 \leq A$ such that $\lambda$ is extendible to $A_1$ and $(A, A_1, \psi)$ is fully ramified for any $\psi \in \text{Irr}(A_1|\lambda)$ (see [4, Theorem 2.7]). As $G$ is irreducible and $\overline{A}$ is a chief factor of $\overline{G}$, we have $A = A_1$ and our claim is verified.

Let $X$ be the preimage in $G$ of socle($\overline{G}$). Take $\overline{A}_1$ and $\overline{A}_2$ minimal normal subgroups of $G$ such that $|\overline{A}_1| \leq |\overline{A}_2|$. Let $\chi$ be the unique irreducible constituent of $\chi^G$. If $\psi$ is an extension of $\lambda$ to $A_1$, then $\text{Irr}(A_1|\lambda) = \{\psi\mu|\psi \in \text{Irr}(A_1/Z)\}$. But $\text{Irr}(A_1|\lambda)$ is also the set of irreducible constituents of $\chi_{A_1}$, and hence $G$ permutes its elements transitively by conjugation.

As socle($\overline{G}$) is abelian then $I_{A_2}(\psi) = I_{A_2}(\psi\mu)$ for every $\mu$ and hence $I_{A_2}(\psi)$ is normal in $G$. Thus $I_{A_2}(\psi) = A_2$ or $I_{A_2}(\psi) = Z$. In the second case the character $\psi^{A_1, A_2}$ is irreducible and its degree is $|\overline{A}_2|$. Hence $|\overline{A}_2|^2 \leq |A_1 A_2 : Z(A_1 A_2)|$. As $Z$ is central in $A_1 A_2$ this forces to $|\overline{A}_1| = |\overline{A}_2|$ and thus $(A_1 A_2, Z, \lambda)$ is fully ramified. As $A_1 A_2$ is a normal nilpotent subgroup of $G$ this contradicts the irreducibility of $G$. Hence $A_2$ leaves invariant each $\psi\mu$ and, as $\psi\mu$ is linear, we have

$$[A_1, A_2] \leq \bigcap_{\mu} \ker(\psi\mu) = \ker \lambda^{A_1} = 1$$

since $\lambda$ is faithful.

Now $X$ is abelian and clearly $\lambda$ is extendible to $X$.

PROOF OF THE THEOREM. As in the Lemma, $G$ is solvable and $\lambda$ is faithful. (i) Let $X$ be the preimage of socle($\overline{G}$) in $G$. If $G$ is not nilpotent then $\lambda$ is extendible to $X$ by the Lemma. Take a prime $p$ satisfying the hypothesis and let $P \in S_p(G)$. Put $\lambda = \prod_q \lambda_q$ where $\lambda_q \in \text{Irr}(Z_q)$, $Z_q \in S_q(Z)$. Now $(P, Z(P), \lambda_p)$ is fully ramified and $Z(P) = Z \cap P$ (see [1, Theorem 2]). Let $\chi_p$ be the unique irreducible constituent of $\lambda^P_p$. Let $\psi_p$ be an extension of $\lambda_p$ to $P \cap N$. Now $\chi_p$ is a constituent of $\psi_p^P$ and $\chi_p(1)^2 \leq |P : X \cap P|^2 = |G : \text{socle}(G)|^2 < |G|_p = |P : Z(P)|$, a contradiction.

(ii) Put $\lambda = \prod_q \lambda_q$, where $\lambda_q \in \text{Irr}(Z_q)$, $Z_q \in S_q(Z)$. If $Q \in S_q(G)$ then $Z(Q) = Q \cap Z$ and $(Q, Z(Q), \lambda_q)$ is fully ramified as in (i). Let $X_q \in S_q(X)$, $q$ a prime. If $\chi_q$ is the unique irreducible constituent of $\lambda_q^Q$ then the elements of $\text{Irr}(X_q \cap Q|\lambda_q)$ are the irreducible constituents of $(\chi_q)_{X_q \cap Q}$. Hence they are conjugate in $Q$. But $\lambda_q$ is invariant in $G$ and $X_q \cap Q$ is normal in $G$. Hence $G$ permutes by conjugation the elements of $\text{Irr}(X_q \cap Q|\lambda_q)$. We conclude that if $\psi_q \in \text{Irr}(X_q \cap Q|\lambda_q)$ then $I_G(\psi_q)$ contains a Hall $q'$-subgroup of $G$.

We may assume that $G$ is not nilpotent and then the Lemma applies. Hence the elements of $\text{Irr}(X_q \cap Q|\lambda_q)$ are linear and $X$ is abelian as we showed. Now let $T$ be a subgroup of $G$ such that $T/X$ is a normal $p$-subgroup of $\overline{G}/X$ for a prime $p$. We show that $O_{p'}(\overline{X}) \leq Z(\overline{T})$. Take $q \neq p$ and $X_q \in S_q(X)$. As $X$ is abelian and $T/X$ is a normal $q'$-subgroup of $G/X$, we have that $T$ leaves invariant each element of $\text{Irr}(X_q \cap Q|\lambda_q)$ by the preceding paragraph. Thus $[T, X_q \cap Q] \leq \ker \lambda_q^{X_q \cap Q} = 1$ since $\lambda_q$ is faithful. Thus our claim is verified and hence $\overline{T}$ is nilpotent. Then $F(\overline{G}/\text{socle}(\overline{G})) \leq F(\overline{G})/\text{socle}(\overline{G})$. As the reverse inclusion is obvious, the first part
of (ii) is verified. To prove the second take a prime $p$ and that $\overline{G}/\text{socle}(G)$ is $p$-closed and let $P$ be a Sylow $p$-subgroup of $G$. Now $P$ is normal in $G$. As $(PZ, Z, \lambda)$ is fully ramified, we deduce that $G$ is nilpotent.

We recall the following definition (see [2, p. 123]): A $p$-group $Q$ is reductive if, for every fully ramified triple $(H, Z, \lambda)$ with $Q$ isomorphic to a Sylow $p$-subgroup of $H/Z$, the irreducible constituent $\chi$ of $\lambda^H$ is fully ramified in $\text{Op}(H)Z$.

In Lemma 4.4 of [2] Gagola showed that every $p$-group of order $p^4$ is reductive. There exist nonreductive $p$-groups at least for $p = 2, 3$, as Example 8.2 of [3] shows. As this example is really complicated and it is not apparent that it might be extended to the remaining primes, we construct a nonreductive $p$-group of order $p^6$ for $p$ odd.

We consider $A_0 = \langle a, b, c, z \rangle \cong (C_p)^4$. By the theory of cyclic extensions there exists an extension $A$ of $A_0$ by $B_0 = \langle \alpha, \beta, \gamma \rangle \cong (C_p)^3$ where $\alpha^p = \beta^p = \gamma^p = 1$, $[\alpha, \beta] = c$, $[\alpha, \gamma] = a$, $[\beta, \gamma] = b$, $[a, \alpha] = [b, \beta]^{-1} = [c, \gamma] = z$, the remaining commutators being trivial.

Let $B$ be the quaternion group of order 8 acting on $A$ as follows: If $e$ and $f$ are generators of $B$ then $a^f = \beta$, $\beta^f = \alpha$, $\gamma^f = \gamma^{-1}$, $a^f = b^{-1}$, $b^f = a^{-1}$, $c^f = c^{-1}$, $z^f = z$ and $e$ acts trivially.

It is an easy check that this action is well defined. Let $G$ be the natural semidirect product of $A$ by $B$. Let $\lambda$ be a nonprincipal character of $A_0/\langle a, b, c \rangle$ and $\mu$ a faithful character of $\langle e \rangle$. Consider $\lambda \times \mu$ as a character of $A_0 \times \langle e \rangle$. Clearly $I_G(\lambda \times \mu) = I_A(\lambda) \cap I_B(\mu)$. It is an easy check that $N_A(\text{Ker } \lambda) = A_0$ and hence $I_A(\lambda) = A_0$. Clearly $I_B(\mu) = \langle e \rangle$. Then $I_G(\lambda \times \mu) = A_0 \times \langle e \rangle$. Thus $(\lambda \times \mu)^G$ is irreducible and $(\lambda \times \mu)^G(1)^2 = |G : Z(G)|$. Hence $G$ is of central type. Now $\text{Op}(G)$ contains $\alpha^{-1}\beta, \gamma$ and $c$. As is normal in $G$, it contains $[\alpha, \gamma]$ and $[\beta, \gamma]$. Thus $|G : \text{Op}(G)Z| = p$ and $(\lambda \times \mu)^G$ is not fully ramified in $\text{Op}(G)Z$.

Using a similar technique it is possible to construct a nonreductive 2-group of order $2^6$.

REFERENCES


DEPARTAMENTO DE ALGEBRA, FACULTAD DE CIENCIAS, UNIVERSIDAD DE ZARAGOZA, 50009 ZARAGOZA, SPAIN

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