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ON BIRKHOFF QUADRATURE FORMULAS

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ABSTRACT. In an earlier work the author has obtained new quadrature formulas (see (1.3)) based on function values and second derivatives on the zeros of $\Pi_n(x)$ as defined by (1.2). The proof given earlier was quite long. The object of this paper is to provide a proof of this quadrature formula which is extremely simple and indeed does not even require the use of fundamental polynomials of $(0,2)$ interpolation.

Introduction. In [**7**] the author obtained some new quadrature formulas based on the zeros

$$1 = x_{1n} > x_{2n} > \cdots > x_{n-1,n} > x_{n,n} = -1 \tag{1.1}$$

of

$$\Pi_n(x) = (1-x^2)P'_{n-1}(x), \tag{1.2}$$

where $P_{n-1}(x)$ denotes the Legendre polynomials of degree $\leq n-1$. More precisely it was proved that the quadrature formula

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx = {} & \frac{3}{n(2n-1)}[f(1)+f(-1)] \\ & + \frac{2(2n-3)}{n(n-2)(2n-1)}\sum_{k=2}^{n-1}\frac{f(x_{kn})}{(P_{n-1}(x_{kn}))^2} \\ & + \frac{1}{n(n-1)(n-2)(2n-1)}\sum_{k=2}^{n-1}\frac{(1-x_{kn}^2)f''(x_{kn})}{(P_{n-1}(x_{kn}))^2}\end{aligned} \tag{1.3}$$

is exact for all polynomials f of degree at most $2n-1$.

The interesting feature of this quadrature formula is that it is based on function values and second derivatives on the zeros of $\Pi_n(x)$ as defined by (1.2). Moreover the above quadrature formula provides the solution of the open problems XXXVI, XXXVII, XXXVIII and XXXIX raised by P. Turán [**6**]. Proof of this quadrature formula was obtained by integrating the fundamental polynomials of $(0,2)$ interpolation investigated earlier by J. Balazs and P. Turán [**1**]. (By the problem of $(0,2)$ interpolation we mean that the values and second derivatives of the interpolatory polynomials are prescribed at the given nodes. For many contributions on Birkhoff quadrature formulas we refer the reader to the interesting monograph of Lorentz et al. [**4**].) As noted in my paper [**7**] the integration of the fundamental polynomials of $(0,2)$ interpolation is complicated and runs in many pages. The main object of this paper is to provide a proof of this quadrature formula which is extremely simple and indeed does not even require the use of fundamental polynomials of $(0,2)$

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interpolation. The ideas of this paper are related to recent work written jointly with Paul Nevai [**5**].

Next, let us denote by $\{P_n^{(\alpha)}\}_{n=0}^{\infty}$ the system of ultraspherical polynomials which are orthogonal in $[-1,1]$ with respect to the weight function $(1-x^2)^\alpha$. We denote by

$$1 > x_{1n}^{(\alpha)} > \cdots > x_{nn}^{(\alpha)} > -1$$

the zeros of $P_n^{(\alpha)}(x)$. We also mention the following quadrature formula valid for all polynomials $f(x)$ of degree $\leq 2n-1$ and based on the zeros of $P_n^{(\alpha)}(x)$ only. It is given by $(\alpha > 0)$

$$\int_{-1}^{1} f(x)(1-x^2)^{\alpha-1}\,dx = \frac{1+2\alpha}{2\alpha}\sum_{k=1}^{n} f(x_{kn}^{(\alpha)})\lambda_{kn}^{(\alpha)} + \frac{1}{4\alpha(1+\alpha)}\sum_{k=1}^{n}(1-(x_{kn}^{(\alpha)})^2)f''(x_{kn}^{(\alpha)})\lambda_{kn}^{(\alpha)}. \tag{1.4}$$

We note that the weight function considered here is $(1-x^2)^{\alpha-1}$ instead of $(1-x^2)^\alpha$, but the nodes $x_{kn}^{(\alpha)}$ are zeros of $P_n^{(\alpha)}(x)$. It turns out that the coefficients of this q.f. are nonnegative for $\alpha > 0$.

2. Proof of quadrature formula (1.3). From integration by parts it follows that for any $f \in c^2[-1,1]$ we have

$$\frac{1}{2}\int_{-1}^{1}(1-x^2)f''(x)\,dx = f(1)+f(-1)-\int_{-1}^{1} f(x)\,dx. \tag{2.1}$$

Multiplying both sides of (2.1) by A_n and rearranging we have (choice of A_n will be made later)

$$\frac{1}{2}A_n\int_{-1}^{1}(1-x^2)f''(x)\,dx = A_n(f(1)+f(-1)) - (1+A_n)\int_{-1}^{1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx. \tag{2.2}$$

Next, consider the following quadrature formula (based on (1.1)) given by

$$\int_{-1}^{1} f(x)\,dx = \frac{2}{n(n-1)}(f(1)+f(-1)) + \frac{2}{n(n-1)}\sum_{k=2}^{n-1}\frac{f(x_{kn})}{P_{n-1}^2(x_{kn})} \tag{2.3}$$

(see [**3**, Chapter IX, Theorem 1, p. 161]). This formula is only exact for $f(x)$ belonging to polynomials of degree $\leq 2n-3$. Applying this quadrature formula we have from (2.2)

$$\begin{aligned}\frac{1}{2}A_n\frac{2}{n(n-1)}\sum_{k=2}^{n-1}\frac{(1-x_{kn}^2)f''(x_{kn})}{P_{n-1}^2(x_{kn})} &= A_n(f(1)+f(-1)) - (1+A_n)\left\{\frac{2}{n(n-1)}(f(1)+f(-1))\right.\\ &\quad\left.+\frac{2}{n(n-1)}\sum_{k=2}^{n-1}\frac{f(x_{kn})}{P_{n-1}^2(x_{kn})}\right\} + \int_{-1}^{1} f(x)\,dx.\end{aligned} \tag{2.4}$$

Obviously (2.4) is exact for $f(x)$ belonging to any polynomial of degree $\leq 2n-3$ for an arbitrary choice of A_n. Our choice of A_n is based on the fact that (2.4) is also exact for polynomials of degree $\leq 2n-2$. Let us put $f(x) = (1-x^2)(P'_{n-1}(x))^2$ and note that it is a polynomial of degree $2n-2$. Therefore, on using (2.4) we obtain

$$\int_{-1}^{1} (1-x^2)(P'_{n-1}(x))^2\,dx = \frac{A_n}{n(n-1)} \sum_{k=2}^{n-1} \frac{(1-x_{kn}^2)^2 2P''_{n-1}(x_k)^2}{P_{n-1}^2(x_k)}. \tag{2.5}$$

Since

$$(1-x_{kn}^2)(P''_{n-1}(x_{kn})) = n(n-1)P_{n-1}(x_{kn}) \tag{2.6}$$

and

$$\int_{-1}^{1} (1-x^2)(P'_{n-1}(x))^2\,dx = \frac{2n(n-1)}{2n-1}, \tag{2.7}$$

it follows that

$$\frac{2n(n-1)}{2n-1} = 2A_n n(n-1) \sum_{k=2}^{n-1} 1 = 2n(n-1)A_n(n-2),$$

$$A_n = \frac{1}{(2n-1)(n-2)}.$$

On substituting this value of A_n in (2.4) we obtain (1.3), exact for polynomials of degree $\leq 2n-2$. But if $f(x)$ is any odd polynomial, (1.3) is obviously valid. Thus we may easily conclude that (1.3) is exact for polynomials of degree $\leq 2n-1$.

Proof of quadrature formula (1.4) is a direct application of Gaussian q.f. and the following identity valid for $f \in c^2[-1,1]$ and $\alpha > 0$:

$$\frac{1}{4\alpha(1+\alpha)} \int_{-1}^{1} (1-x^2)f''(x)(1-x^2)^\alpha\,dx + \frac{(1+2\alpha)}{2\alpha} \int_{-1}^{1} f(x)(1-x^2)^\alpha\,dx$$
$$= \int_{-1}^{1} f(x)(1-x^2)^{\alpha-1}\,dx, \qquad \alpha > 0.$$

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