SIMILARITY-ININVARIANT CONTINUOUS
FUNCTIONS ON $\mathcal{L}(\mathcal{H})$

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ABSTRACT. Let $f: \mathcal{L}(\mathcal{H}) \to X$ be a continuous function from the algebra of all bounded linear operators acting on a complex infinite dimensional Hilbert space $\mathcal{H}$ into a $T_1$-topological space $X$. If $f(WAW^{-1}) = f(A)$ for all $A$ in $\mathcal{L}(\mathcal{H})$ and all invertible $W$, then $f$ is a constant function. The same result is true for a function $f$ satisfying the above conditions defined on a connected open subset of $\mathcal{L}(\mathcal{H})_0 = \{T \in \mathcal{L}(\mathcal{H}) : T$ has no normal eigenvalues$\}$.

1. Introduction. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all (bounded linear) operators acting on a complex Hilbert space $\mathcal{H}$. The classical spectral functions ($\sigma$ (spectrum), $sp$ (spectral radius), $\sigma_e$ (Calkin essential spectrum), $\sigma_{rec}$ (Wolf essential spectrum, i.e., the complement in the complex plane $C$ of the semi-Fredholm domain), etc.), mapping $\mathcal{L}(\mathcal{H})$ (endowed with the norm-topology) into the space $X$ of all compact subsets of $C$ (Hausdorff metric), or $X = \mathbb{R}$ (the real interval $[0, \infty)$ (the usual topology) have a very erratic behavior.

Indeed, if $\mathcal{H}$ is infinite dimensional, all these functions, and the uncountably many analyzed in [3], with the single exception of the spectral radius, are continuous on a dense subset of $\mathcal{L}(\mathcal{H})$, and discontinuous on another dense subset of $\mathcal{L}(\mathcal{H})$!

The spectral radius, on the other hand, is continuous on an open dense subset of $\mathcal{L}(\mathcal{H})$, but not everywhere. (We can also find certain “natural” spectral functions which are discontinuous everywhere; see the above reference.)

Is this behavior a peculiarity of our particular functions? Or, is it possible to construct some “natural” spectral function which is continuous everywhere? The answer is: NO. The existence of discontinuities for all these functions is in the nature of things, not a peculiarity of the special functions considered in the literature, and the deep reason is that the spectral functions are similarity-invariant; that is, they take the same value on all the elements of the similarity orbit

$$S(T) = \{WTW^{-1} : W \in \mathcal{G}(\mathcal{H})\}$$

of an operator $T$ in $\mathcal{L}(\mathcal{H})$. (Here $\mathcal{G}(\mathcal{H})$ denotes the group of all invertible operators.)

THEOREM 1. Let $f: \mathcal{L}(\mathcal{H}) \to X$ be a continuous function defined on $\mathcal{L}(\mathcal{H})$ ($\mathcal{H}$ a complex separable infinite dimensional Hilbert space) with values on a $T_1$-topological space $X$.

Suppose that $f(WTW^{-1}) = f(T)$ for all $T \in \mathcal{L}(\mathcal{H})$, and all $W \in \mathcal{G}(\mathcal{H})$. Then $f$ is a constant function.

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The special case of the spectral radius (continuous on an open dense subset) can be explained as a sort of finite dimensional phenomenon of the following type: Given an operator $A$, acting on a complex infinite dimensional Banach space $X$, and $\varepsilon > 0$, there exists $B \in \mathcal{L}(X)$ such that $A - B$ is a finite rank operator, $\|A - B\| < \varepsilon$, and $B$ has a normal eigenvalue $\mu$ (that is, $\mu$ is an isolated point of the spectrum, $\sigma(B)$, of $B$, and the Riesz spectral subspace $X(\{\mu\}; B)$ corresponding to the clopen subset $\{\mu\}$ of $\sigma(B)$ is finite dimensional; see [4]).

Given $B$ and $\mu$ as above, there exist $\eta > 0$ and $\delta > 0$ such that, if $C \in \mathcal{L}(X)$ and $\|B - C\| < \eta$, then $\sigma(C)$ does not intersect the circle $\gamma(\mu; \delta) = \{\lambda \in \mathbb{C}: |\lambda - \mu| = \delta\}$, $\sigma = \sigma(C) \cap \{\lambda: |\lambda - \mu| < \delta\} \neq \emptyset$, and the Riesz spectral subspace $\mathcal{X}(\sigma; C)$ corresponding to the clopen subset $\sigma$ of $\sigma(C)$ is isomorphic to $X(\{\mu\}; B)$.

For practical purposes, we can directly assume that $\mathcal{X}(\{\mu\}; B)$ and $\mathcal{X}(\sigma; C)$ are identified with $\mathbb{C}^n$ (where $n = \text{dim} \mathcal{X}(\{\mu\}; B)$). Let $f$ be a similarity-invariant function defined on $\mathcal{L}(\mathbb{C}^n)$, with values in a $T_1$-space $X$ (for instance, $f(T) = \text{trace}(T)$, or $f(T) = \sigma(T)$, or $f(T) = \text{sp}(T)$, etc.); then we can define

$$g: \{C \in \mathcal{L}(X): \|B - C\| < \eta\} \to X$$

(by using the above identification) via $g(C) = f(C|\mathcal{X}(\sigma; C))$.

It is easily seen that $g$ is continuous, and $g(WCW^{-1}) = g(C)$ for all $C \in \mathcal{L}(X)$ and all $W$ in $\mathcal{G}(X)$ such that $\|B - C\| < \eta$, $\|B - WCW^{-1}\| < \eta$.

Let

$$\mathcal{L}(X)_0 = \{T \in \mathcal{L}(X): \sigma_0(T) = \emptyset\},$$

where $\sigma_0(T)$ denotes the set of all normal eigenvalues of $T$. It is well known that $\mathcal{L}(X)_0$ is a closed nowhere dense subset of $\mathcal{L}(X)$ [4] (see also [1, Proposition 11.30]). We have the following result:

**PROPOSITION 2.** Let $X$ be a complex infinite dimensional Banach space. For each $B$ in the open dense subset $\mathcal{L}(X)\setminus \mathcal{L}(X)_0$ (of $\mathcal{L}(X)$), there exists a nonconstant continuous similarity-invariant function $g$ defined on some neighborhood of $B$.

(Compare this result and its proof with Proposition 8 of [3], on the set of points of continuity of the spectral radius function.)

Similarity-invariant continuous functions from $\mathcal{L}(\mathbb{C}^n)$ into $\mathbb{C}$ admit a very simple characterization. The reader can find a short proof of this characterization in the recently published note of R. Koch [6]. (Clearly, $\mathbb{C}$ can be replaced by any $T_1$-space in the above-mentioned note. The author is deeply indebted to Professor Abraham Sinkov for calling his attention to Koch’s article.)

In the case when $X$ is an infinite dimensional Hilbert space, Proposition 2 is the best possible result on these lines. Indeed, we have

**THEOREM 3.** Let $H$ be a complex separable infinite dimensional Hilbert space. Let $\Omega$ be a connected open (in the relative norm-topology) subset of $\mathcal{L}(H)_0$. If $f: \Omega \to X$ is a continuous function defined on $\Omega$ with values on a $T_1$-space $X$ such that $f(WTW^{-1}) = f(T)$ for all $T$ in $\mathcal{L}(H)_0$ and all $W$ in $\mathcal{G}(H)$ such that $T$, $WTW^{-1} \in \Omega$, then $f$ is a constant function.

**2. Proof of Theorem 1.** We shall need two results on approximation of operators. Recall that $\sigma_c(T) = \sigma(\pi(T))$ $(T \in \mathcal{L}(H))$, where $\pi: \mathcal{L}(H) \to A(H) = \mathcal{L}(H)/\mathcal{K}(H)$ is the canonical projection onto the Calkin algebra. (Here $\mathcal{K}(H)$ denotes the ideal of all compact operators.)
In what follows, \( \mathcal{L}(\mathcal{H})_u \) will denote the family of all those operators \( T \) in \( \mathcal{L}(\mathcal{H}) \) with the following property: If \( \mu \) is an isolated point of \( \sigma_e(T) \) and

\[
f_{\mu,k}(\lambda) = \begin{cases} 
(\lambda - \mu)^k & \text{on some neighborhood of } \mu, \\
0 & \text{on some neighborhood of } \sigma_e(T) \setminus \{\mu\},
\end{cases}
\]

then \( f_{\mu,k}(\pi(T)) \neq 0 \) (\( k = 1, 2, \ldots \); \( f_{\mu,k}(\pi(T)) \) is defined, as usual, via functional calculus).

The first result is a weak version of Theorem 9.1 of [1].

**Proposition 4.** Let \( T \in \mathcal{L}(\mathcal{H})_u \). Then the closure \( S(T)^- \), of \( S(T) \), coincides with the set of all those operators \( A \) in \( \mathcal{L}(\mathcal{H}) \) satisfying the conditions:

1. \( \sigma_0(A) \subset \sigma_0(T) \) and \( \dim \mathcal{H}(\{\lambda\}; A) = \dim \mathcal{H}(\{\lambda\}; T) \) for all \( \lambda \) in \( \sigma_0(A) \);
2. each component of \( \sigma_{\text{tr}}(A) \) intersects \( \sigma_e(T) \);
3. the semi-Fredholm domain \( \rho_{\text{SFD}}(A) \) [2, 5], of \( A \), is a subset of \( \rho_{\text{SFD}}(T) \), and \( \text{ind}(\lambda - A) = \text{ind}(\lambda - T) \) for all \( \lambda \in \rho_{\text{SFD}}(A) \);
4. \( \min\{\text{nul}(\lambda - A)^k, \text{nul}(\lambda - A)^*k\} \geq \min\{\text{nul}(\lambda - T)^k, \text{nul}(\lambda - T)^*k\} \) for all \( \lambda \in \rho_{\text{SFD}}(A) \) and all \( k = 1, 2, \ldots \).

Furthermore, if

\[
S(\pi(T)) = \{\pi(W)\pi(T)\pi(W)^{-1}: \pi(W) \in \mathcal{C}[\mathcal{L}(\mathcal{H})]\},
\]

then

\[
S(\pi(T))^- = \{\pi(A) \in \mathcal{A}(\mathcal{H}): A \text{ satisfies (S) and (F)}\}.
\]

In particular, \( \mathcal{L}(\mathcal{H})_u \) is dense in \( \mathcal{L}(\mathcal{H}) \), and \( \pi[\mathcal{L}(\mathcal{H})_u] \) is dense in \( \mathcal{A}(\mathcal{H}) \).

The second result is a corollary of the first one, and a standard approximation result (see, e.g., [2, Corollary 3.50]).

**Proposition 5.** Let \( A \in \mathcal{L}(\mathcal{H}) \), let \( \varepsilon > 0 \) and let \( M \) be a normal operator such that

\[
\sigma(M) = \{\lambda \in \mathbb{C}: \text{dist}[\lambda, \sigma_{\text{tr}}(A)] \leq \varepsilon/2\}
\]

There exists \( A' \) similar to \( A \oplus M \) such that \( \|A - A'\| < \varepsilon \). Furthermore, if \( A \in \mathcal{L}(\mathcal{H})_u \), then \( A' \in S(A)^- \).

Let \( f \) and \( X \) be as in Theorem 1. Since the singletons are closed in \( X \), and \( f \) is continuous and similarity-invariant, we infer that \( f[S(T)^-] \) is constant for each \( T \) in \( \mathcal{L}(\mathcal{H}) \).

Let \( A, B \in \mathcal{L}(\mathcal{H})_u \). Then Proposition 4 implies that \( S(A)^- \cap S(B)^- \neq \emptyset \). For instance, the intersection contains every normal operator \( M \) such that \( \sigma(M) = \{\lambda \in \mathbb{C}: |\lambda| \leq \text{max}[\text{sp}(A), \text{sp}(B)]\} \). It follows that \( f(A) = f(M) = f(B) \), and therefore \( f \) is constant on \( \mathcal{L}(\mathcal{H})_u \).

Since, by Proposition 4, \( \mathcal{L}(\mathcal{H})_u \) is dense in \( \mathcal{L}(\mathcal{H}) \), we conclude that \( f \) is a constant function.

**Remark.** Theorem 1 remains true (by the same proof) if we merely assume that \( f \) is only defined on some ball about \( 0 \in \mathcal{L}(\mathcal{H}) \).

3. **Proof of Theorem 3.** Let \( A \in \Omega \) and let \( \varepsilon > 0 \) be small enough to guarantee that the intersection \( \mathcal{B}(A; \varepsilon) \) of the ball of radius \( \varepsilon \) about \( A \) with \( \mathcal{L}(\mathcal{H})_0 \) is included in \( \Omega \). Let the normal operator \( M \), and let the operator \( A' \) similar to \( A \oplus M \) be defined as in Proposition 5 (so that \( A' \in \mathcal{B}(A; \varepsilon) \)).
By Proposition 4, we can find $A'' \in \mathcal{B}(A; \varepsilon) \cap \mathcal{L}$(H) such that $A, A' \in \mathcal{S}(A'')$. By using the stability properties of the index and a standard compactness argument, we can find $\eta, 0 < \eta < \varepsilon/2$, such that $\rho_{\psi^{-1}}(B) \supset \rho_{\psi^{-1}}(A') = \rho_{\psi^{-1}}(A) \setminus \sigma(M)$, and $\text{ind}(\lambda - B) = \text{ind}(\lambda - A) (= \text{ind}(\lambda - A'))$ for all $\lambda \in \rho_{\psi^{-1}}(A')$ and all $B$ in $\mathcal{L}$(H) such that $\|A - B\| < \eta$.

We deduce, as in the proof of Theorem 1, that $f(A) = f(A'') = f(A') = f(C)$ for all $C \in \mathcal{B}(A; \eta) \cap \mathcal{L}$(H) such that $\min\{\text{nul}(\lambda - C), \text{nul}(\lambda - C)^*\} = 0$ for all $\lambda \in \rho_{\psi^{-1}}(B)$. (Observe that $A' \in \mathcal{S}(C)^{-1}$.)

By Proposition 4, every $B$ in $\mathcal{B}(A; \eta)$ belongs to $\mathcal{S}(C)^{-1}$ for some $C$ as above. But in this case $f(B) = f(C) = f(A)$, so that $f$ is constant on $\mathcal{B}(A; \eta)$.

Since $\Omega$ is connected, we conclude that $f$ is constant on $\Omega$. \hfill \Box

Minor modifications of the same argument yield the following analogous result for the Calkin algebra:

**Theorem 6.** A continuous similarity-invariant function $h$ defined on an open connected subset $\Phi$ of $\mathcal{A}(\mathcal{H})$, with values on a $T_1$-space, is necessarily constant on $\Phi$.

**Remark.** Theorems 1, 3, and 6 remain true if $\mathcal{H}$ is nonseparable. Indeed, if $T \in \mathcal{L}$(H) (H nonseparable), we can always find a separable subspace $\mathcal{R}$ reducing $T$ such that $\sigma(T|\mathcal{R}) = \sigma(T)$, $\sigma_0(T|\mathcal{R}) = \sigma_0(T)$, $\mathcal{H}((\lambda); T|\mathcal{R}) = \mathcal{H}((\lambda); T)$ for all $\lambda \in \sigma_0(T)$, $\sigma_c(T|\mathcal{R}) = \sigma_c(T)$, $\sigma_{\text{irr}}(T|\mathcal{R}) = \sigma_{\text{irr}}(T)$, and $\text{ind}(\lambda - T|\mathcal{R}) = \text{ind}(\lambda - T)$ for all $\lambda \in \rho_{\psi^{-1}}(T)$. Now the results can be proved exactly as in the separable case, by modifying $A, B,\text{ etc.}$, only on a suitable separable subspace reducing both operators. The details are left to the reader.

**References**


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