ON WEAKLY COMPACT OPERATORS ON SPACES OF VECTOR VALUED CONTINUOUS FUNCTIONS

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Abstract. Let $K$ and $S$ be compact Hausdorff spaces and $\theta$ a continuous function from $K$ onto $S$. Then for any Banach space $E$ the map $f \mapsto f \circ \theta$ isometrically embeds $C(S, E)$ as a closed subspace of $C(K, E)$. In this note we prove that when $E'$ has the Radon-Nikodym property, every weakly compact operator on $C(S, E)$ can be lifted to a weakly compact operator on $C(K, E)$. As a consequence, we prove that the compact dispersed spaces $K$ are characterized by the fact that $C(K, E)$ has the Dunford-Pettis property whenever $E$ has.

For a Banach space $E$ and a compact Hausdorff space $K$, $C(K, E)$ will denote the Banach space of all $E$-valued continuous functions on $K$ under the supremum norm. We shall identify, as usual, the topological dual of $C(K, E)$ with the space $\text{rcabv}(\text{Bo}(K), E')$ of the regular, countably additive Borel measures on $K$ of bounded variation, with values in the topological dual $E'$ of $E$, endowed with the variation norm [3, Theorem 2.2].

The notation and terminology used and not defined can be found in [6, 10, or 11].

Let $K$ and $S$ be compact Hausdorff spaces, and let $\theta: K \rightarrow S$ be an onto continuous function. For any Banach space $E$, the map $f \mapsto J_\theta(f) = f \circ \theta$ is then a linear isometry which embeds $C(S, E)$ as a closed subspace of $C(K, E)$. The following result establishes that, in some cases, every weakly compact operator on $C(S, E)$ can be lifted to a weakly compact operator on $C(K, E)$.

Theorem 1. Let $K$, $S$ and $\theta$ be as above, and let $E$ be a Banach space such that $E'$ has the Radon-Nikodym property. Then, if $F$ is a Banach space and $U: C(S, E) \rightarrow F$ is a weakly compact operator, there exists a weakly compact operator $\bar{U}: C(K, E) \rightarrow F$ so that $\|\bar{U}\| = \|U\|$ and $\bar{U} \circ J_\theta = U$.

Proof. Let $D$ be the closed unit ball of $F'$. Then $U'(D)$ is weakly compact in $\text{rcabv}(\text{Bo}(S), E')$, and so there is a control measure $\lambda$ for $U'(D)$ [3, Proposition 3.1; 6, 1.2.4]; i.e., a positive Radon measure on $S$ such that $\lim_{\lambda(B) \rightarrow 0} \|m(B)\| = 0$, uniformly in $m \in U'(D)$. Because $E'$ has the Radon-Nikodym property, for each $x' \in D$ there exists a Bochner integrable density $g_{x'} \in L^1(S, \lambda, E')$ for $U'(x')$. Recalling that the map $g \mapsto m_g$, where $m_g(f) = \int \langle f, g \rangle \, d\lambda$ for each $f$ in $C(S, E)$, is an isometry from $L^1(S, \lambda, E')$ into $\text{rcabv}(\text{Bo}(S), E')$, it follows that $\{g_{x'}: x' \in D\}$ is weakly compact in $L^1(S, \lambda, E')$.

Received by the editors February 14, 1985 and, in revised form, April 21, 1985.

1980 Mathematics Subject Classification. Primary 46E40, 46G10; Secondary 47B38.

Supported in part by CAICYT grant 0338-84.
Now let \( \mu \) be a positive Radon measure on \( K \) such that \( \theta(\mu) = \lambda \) [11, Chapter I, Theorem 12]. The map
\[
L^1(S, \lambda, E') \ni g \mapsto g \circ \theta \in L^1(K, \mu, E')
\]
is a linear isometry, and so \( \{ g_{x'} \circ \theta : x' \in D \} \) is a weakly compact subset of \( L^1(K, \mu, E') \).

For \( f \in C(K, E) \) and \( x' \in F' \) let us define
\[
\langle U(f), x' \rangle = \int \langle f, g_{x'} \circ \theta \rangle \, d\mu.
\]

\( \overline{U}(f) \) clearly belongs to the algebraic dual of \( F' \). But
\[
|\langle \overline{U}(f), x' \rangle| \leq \| f \| \int \| g_{x'} \circ \theta \| \, d\mu = \| f \| \| U'(x') \|
\]
\[
\leq \| U \| \| f \| \| x' \|
\]
which proves that \( \overline{U}(f) \in F'' \), and so \( \overline{U} \) is a continuous linear operator from \( C(K, E) \) to \( F'' \) with \( \| \overline{U} \| \leq \| U \| \). Besides, \( \overline{U} \circ J_\theta = U \), and then \( \| \overline{U} \| = \| U \| \).

Let \( (x'_i)_{i \in I} \subset D \) be a weak* null net. Then \( (U'(x'_i))_{i \in I} \) is relatively weakly compact and weak* convergent to \( 0 \). It follows that it is weakly null, and so \( (g_{x'} \circ \theta)_{i \in I} \) converges weakly to \( 0 \) in \( L^1(K, \mu, E') \). Therefore,
\[
\lim_{i \in I} \langle \overline{U}(f), x'_i \rangle = 0.
\]
This shows that the restriction of \( \overline{U}(f) \) to \( D \) is weak* continuous, and so, by Grothendieck's theorem [9, 3.11.4], \( \overline{U}(f) \) belongs to \( F \) for every \( f \) in \( C(K, E) \).

Finally, let us note that \( \overline{U}(D) = \{ g_{x'} \circ \theta : x' \in D \} \) (with the canonical identification of \( L^1(K, \mu, E') \) as a closed subspace of \( rcabv(Bo(K), E') \)), and so \( \overline{U} \), and consequently \( \overline{U} \), is weakly compact.

**Corollary.** Under the assumptions of Theorem 1, if \( (x'_n) \) is a weakly convergent sequence in \( rcabv(Bo(S), E') \), there exists a weakly convergent sequence \( (y'_n) \) in \( rcabv(Bo(K), E') \) such that \( J_\theta(y'_n) = x'_n \) for every \( n \).

**Proof.** We can assume that \( (x'_n) \) is weakly null. The operator \( U : C(S, E) \to c_0 \) defined by \( U(f) = (\langle f, x'_n \rangle) \) is then weakly compact. According to Theorem 1, there is a weakly compact operator \( \overline{U} : C(K, E) \to c_0 \) such that \( \overline{U} \circ J_\theta = U \). If \( \overline{U}(f) = (\langle f, y'_n \rangle) \) for \( f \) in \( C(K, E) \), the sequence \( (y'_n) \) fulfills the requirements of the corollary.

Recall that a Banach space \( E \) is said to have the Dunford-Pettis property (D.P.P.) if every weakly compact operator on \( E \) sends weakly compact sets into norm compact ones. This property was introduced by A. Grothendieck in his important paper [8] and has been intensively studied (see [5]). The long-standing open question of whether \( C(K, E) \) has the D.P.P. if \( E \) has, was answered by M. Talagrand [12], who built a Banach space \( T \) such that

1. \( T \) and \( T' \) have unconditional basis.
2. \( T' \) is a Schur space; in particular, \( T \) and \( T' \) have the D.P.P.
3. \( C([0, 1], T) \) does not have the D.P.P.
However, if $K$ is a dispersed compact space (i.e., a space which does not contain any perfect set; see [10, §5]), it is known that $C(K, E)$ has the D.P.P. if $E$ has (see [4 and 7]). Now we can prove that this property indeed characterizes the compact dispersed spaces.

**Theorem 2.** Let $K$ be a compact Hausdorff space. The following properties are equivalent:

(a) $K$ is dispersed.

(b) If $E$ is a Banach space with the D.P.P., so is $C(K, E)$.

(c) If $T$ is the Talagrand space, $C(K, T)$ has the D.P.P.

**Proof.** That (a) implies (b) is known (see [4, Theorem 4; 7, Theorem 13]) and (b) $\Rightarrow$ (c) is obvious. Let us suppose that $K$ is not dispersed. Then there is a continuous onto function $\theta: K \rightarrow [0, 1]$ [10, §2.4.2]. According to Talagrand’s result, $C([0, 1], T)$ does not have the D.P.P., and so there is a Banach space $F$, a weakly compact operator $U$ from $C([0, 1], T)$ to $F$ and a weakly compact subset $H$ of $C([0, 1], T)$ such that $U(H)$ is not norm compact. Because $T'$ is separable, it has the Radon-Nikodým property [6, III.3.1]. By Theorem 1, there is a weakly compact operator $\overline{U}$ from $C(K, T)$ into $F$ such that $\overline{U} \circ J_\theta = U$. But then $J_\theta(H)$ is weakly compact and $\overline{U}(J_\theta(H)) = U(H)$ is not norm compact. This shows that $C(K, T)$ does not have the D.P.P., and concludes the proof.

**Additional Remark.** After the writing of this paper, some other extension theorems for operators on $C(S, E)$ have been obtained in [2]. By using the methods of this last paper (essentially the choice of a weak* density instead of a Bochner one), one can prove that Theorem 1 is true without assuming the Radon-Nikodým property on $E'$. For details, we refer to [2].

We should also mention that other characterizations of compact dispersed spaces in terms of operators on spaces of continuous vector valued functions can be seen in [1].

**References**


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