PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 2, June 1986

AN ANSWER TO A QUESTION OF M. NEWMAN ON MATRIX COMPLETION

L. N. VASERSTEIN[1]

ABSTRACT. Let R be a principal ideal ring, A a symmetric t-by-t matrix over R, B a t-by-$(n-t)$ matrix over R such that the t-by-n matrix (A, B) is primitive. Newman [**2**] proved that (A, B) may be completed (as the first t rows) to a symmetric n-by-n matrix of determinant 1, provided that $1 \leq t \leq n/3$. He showed that the result is false, in general, if $t = n/2$, and he asked to determine all values of t such that $1 \leq t \leq n$ and the result holds. It is shown here that these values are exactly t satisfying $1 \leq t < n/2$.

Moreover, the result is proved for a larger (than the principal ideal rings) class of commutative rings, namely, for the rings satisfying the second stable range condition of Bass [**1**].

Also, it is observed that Theorems 2 and 3 of [**2**, p. 40] proved there for principal ideal rings are true for this larger class of rings, as well as the basic result of [**2**, p. 39].

Statement of results. In this paper, "ring" means "commutative associative ring with 1".

We use the notation $\mathrm{sr}(R)$ [**3**, **7**]. In particular, $\mathrm{sr}(R) \leq 1$ means that the following condition (the first stable range condition of Bass [**1**]) holds:

($\mathrm{sr}(R) \leq 1$). If a_1, a_2 are in R and $a_1R + a_2R = R$, then $(a_1 + a_2c)R = R$ for some c in R.

This condition was discussed in detail in [**6**], where many examples are given. A. Roy showed me principal ideal rings with $\mathrm{sr}(R) \leq 1$ which are not semilocal. B. Magurn showed me Dedekind rings R with $\mathrm{sr}(R) \leq 1$ which are not principal ideal rings. The following theorem gives more examples (including all Boolean rings R with 1).

THEOREM 1. *Suppose that there is a primitive polynomial $f(x)$ in one variable x with integral coefficients such that $f(r) = 0$ for every element r of a ring R. Then* $\mathrm{sr}(R) \leq 1$.

The second stable range condition, $\mathrm{sr}(R) \leq 2$, reads as follows:

($\mathrm{sr}(R) \leq 2$). If a_1, a_2, a_3 are in R and $a_1R + a_2R + a_3R = R$, then $(a_1 + a_3c_1)R + (a_2 + a_3c_2)R = R$ for some c_1, c_2 in R.

If R is Noetherian of finite Krull dimension $\dim(R)$, then $\mathrm{sr}(R) \leq 1 + \dim(R)$, by Bass [**1**]. In particular, $\mathrm{sr}(R) \leq 1 + \dim(R) \leq 1 + 1 = 2$ for each principal ideal ring R.

Here are more examples of rings R with $\mathrm{sr}(R) \leq 2$: The ring of integers in any number field or, more generally, any subring R with 1 of any global field; any R whose space of maximal ideals is Noetherian of dimension ≤ 1; any finitely

Received by the editors June 4, 1985.
1980 *Mathematics Subject Classification*. Primary 18F25, 15A33.
[1]Supported in part by the National Science and Guggenheim Foundations.

generated algebra R over any finite field with $\dim(R) \leq 2$ (see [**7**]); the ring of real continuous functions on an interval or, more generally, on any topological space of dimension ≤ 1 (see [**3**]); the ring of all continuous complex functions on a topological space of dimension ≤ 3 (see [**3**]).

For any ring R and any natural numbers t and n, let $M_{t,n}R$ denote the set of all t-by-n matrices over R. A matrix in $M_{t,n}R$ is said to be *unimodular* if it has a left (when $t \leq n$) or right (when $t > n$) inverse in $M_{n,t}R$. This is equivalent for this matrix to be *primitive*, i.e. the ideal of R generated by all maximal minors to be R.

Let $\mathrm{GL}_n R$ be the group of unimodular (i.e. invertible) matrices in $M_{n,n}R$, and let $\mathrm{SL}_n R$ denote the subgroup of matrices with determinant 1.

THEOREM 2. *Let R be a ring, and let t and n be natural numbers such that $t \leq n$. If $\mathrm{sr}(R) \leq n - 2t + 1$ and $t < n/2$, then*

$(R_{t,n})$. *For every symmetric matrix $A = A^T$ in $M_{t,t}$ and every matrix B in $M_{t,n-t}R$ such that the matrix (A, B) in $M_{t,n}R$ is unimodular, there is a symmetric matrix in $\mathrm{SL}_n R$ whose first t rows form the given matrix (A, B).*

THEOREM 3. *Let R be a ring, and let t and n be natural numbers such that $t \leq n$. Then*

(a) *when $t = n$, the statement $(R_{t,n})$ above is true if and only if the group $\mathrm{GL}_1 R$ is trivial, i.e 1 is the only unit of R;*

(b) *when $t = n/2$ is odd, $(R_{t,n})$ is true if and only if $r^4 = r^2$ for all r in R and $2R = 0$;*

(c) *when $t = n/2$ is even, $(R_{t,n})$ is true if and only if $(r^2 - 2r)(r^2 - 1) = 0$ for all r in R;*

(d) *when $n/2 < t < n$, $(R_{t,n})$ is true if and only if $r^2 = r$ for all r in R, i.e. R is a Boolean ring.*

Combining Theorems 2 and 3 we obtain the following complete answer to the open question (c) of [**2**, p. 45].

COROLLARY. *If $\mathrm{sr}(R) \leq 2$ (for example, R is a principal ideal ring) then the statement $(R_{t,s})$ holds, provided that $1 \leq t < n/2$. If $(r-2)(r^4 - r^2) \neq 0$ for some r in R, then the statement $(R_{t,n})$ never holds for $n/2 \leq t < n$.*

However, Theorem 3 justifies a modification of the question of Newman for $t \geq n/2$. For example, $(R_{n,n})$ becomes true if we replace $\mathrm{SL}_n R$ in it by $\mathrm{GL}_n R$. The following theorem answers the modified question for all t.

THEOREM 4. *The statement $(R_{t,n})$ with $\mathrm{SL}_n R$ replaced by $\mathrm{GL}_n R$ is true for all t and n such that $t \leq n$ if and only if $\mathrm{sr}(R) \leq 1$.*

REMARK. Let $\mathrm{sr}(R) \leq 2$. Then, by [**4**, **5**], for any natural numbers t and n such that $t \leq n$ and any matrix C in $M_{t,t}R$, the symplectic group

$$\mathrm{Sp}_{2n}R = \left\{ D \in \mathrm{SL}_{2n}R \colon D^T \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} D = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\}$$

acts transitively on the matrices (A, B) in $M_{t,2n}R$ with A, B in $M_{t,n}R$ and $AB^T - BA^T = C$.

Taking here $C = 0$ and $t = 1$ (respectively, $t = n$), we obtain Theorem 2 (resp., Theorem 3) of [**2**], extended from principal ideal rings R to any R with $\mathrm{sr}(R) \leq 2$.

PROOF OF THEOREM 1. Let d be the degree of the polynomial $f(x)$. Then for any maximal ideal m of R the polynomial vanishes on the field R/m, hence $\mathrm{card}(R/m) \leq d$. Therefore $r^{d!} \equiv r \pmod{m}$ for every element r in R and any maximal ideal m of R.

Let (a_1, a_2) be an arbitrary unimodular row in $M_{1,2}R$. We set $c = a_1^{d!} - 1$. The claim is that $(a_1 + a_2c)R = R$. It suffices to show that the left-hand side is not contained in any maximal ideal m of R. Passing to the factor ring R/m, we see that is suffices to prove the claim in the case when R is a field. In this case,

$$a_1 + a_2c = \begin{cases} a_1 & \text{if } a \neq 0 \text{ (because then } c = 0), \\ a_2 & \text{if } a_1 = 0 \text{ (because then } c = -1). \end{cases}$$

So $a_1 + a_2c \neq 0$, hence $(a_1 + a_2c)R = R$.

PROOF OF THEOREM 2. We will use the following two lemmas:

LEMMA 1 (SEE [**3**]). *For any natural numbers d, s, t the condition $\mathrm{sr}(R) \leq d$ is equivalent to the following condition: For any matrices A in $M_{s,d+s-1}R$ and B in $M_{s,t}R$ such that (A, B) in $M_{s,d+s-1+t}R$ is unimodular there is a matrix C in $M_{t,rd+s-1}R$ such that the matrix $A + BC$ in $M_{s,d+s-1}R$ is unimodular.*

LEMMA 2 (SEE [**1**, **5**, **7**]). *For any s, t with $t \geq s + \mathrm{sr}(R)$, the group SL_tR acts transitively on the set of unimodular matrices in $M_{s,t}R$; if $\mathrm{sr}(R) \leq 2$, then the condition $t \geq s + \mathrm{sr}(R)$ above can be replaced by $t \geq s + 1$.*

REMARK. Lemma 2 can be easily proved by induction on s. After this, it is not hard to prove Lemma 1 also by induction on s. Of course, Lemma 2 is a trivial consequence of Lemma 1. Lemma 2 generalizes the basic result of [**2**, p. 39].

Now we are ready to prove Theorem 2. Let $\mathrm{sr}(R) \leq n - 2t + 1$ and let A and B be as in $(R_{t,n})$.

Case 1. B has the form $(1_t, 0)$, where 1_t is the identity matrix in SL_tR and 0 is the zero matrix in $M_{t,n-2t}R$. Then for any matrix $X = X^T$ in $M_{n-2t,n-2t}R$ the matrix

$$Y = \begin{pmatrix} A & 1_t & 0 \\ 1_t & 0 & 0 \\ 0 & 0 & X \end{pmatrix} \quad \text{in } M_{n,n}R$$

is symmetric, its first t rows form the given matrix $(A, B) = (A, 1_t, 0)$, and $\det(Y) = (-1)^t \det(X)$. Choosing X with $\det(X) = (-1)^t$ (a diagonal matrix will do), we complete our proof in Case 1.

Case 2. B is unimodular. Then by Lemma 2 there is a matrix C in $\mathrm{SL}_{n-t}R$ such that $BC = (1_t, 0)$. By Case 1, there is a symmetric matrix Y in SL_nR of the form $Y = \left(\begin{smallmatrix} A & BC \\ * & * \end{smallmatrix}\right)$. Then the symmetric matrix

$$\begin{pmatrix} 1_t & 0 \\ 0 & C^{-1} \end{pmatrix}^T Y \begin{pmatrix} 1_t & 0 \\ 0 & C^{-1} \end{pmatrix} \quad \text{in } \mathrm{SL}_nR$$

is of the form $\left(\begin{smallmatrix} A & B \\ * & * \end{smallmatrix}\right)$.

General case. By Lemma 1, there is a matrix D in $M_{t,n-t}R$ such that $B+AD$ in $M_{t,n-t}R$ is unimodular. By Case 2, there is a symmetric matrix Y in $\mathrm{SL}_n R$ of the form $Y = \left(\begin{smallmatrix} A & B+AD \\ * & * \end{smallmatrix}\right)$. Then the symmetric matrix

$$\begin{pmatrix} 1_t & 0 \\ -D^T & 1_{n-t} \end{pmatrix} Y \begin{pmatrix} 1_t & -D \\ 0 & 1_{n-t} \end{pmatrix} \quad \text{in } \mathrm{SL}_n R$$

is of the form $\left(\begin{smallmatrix} A & B \\ * & * \end{smallmatrix}\right)$.

PROOF OF THEOREM 3. Part (a) is evident, so let $n/2 \le t < n$.

LEMMA 3. *The condition $(R_{t,2t})$ implies that $b^2 \equiv (-1)^t \pmod{aR}$ for every unimodular row (a,b) in $M_{1,2}R$.*

PROOF. We set $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & a \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1_{t-1} & 0 \\ 0 & b \end{smallmatrix}\right)$ (both in $M_{t,t}R$; when $t=1$, $(A,B) = (a,b)$). Let Y be a symmetric matrix in $\mathrm{SL}_{2t}R$ of the form $Y = \left(\begin{smallmatrix} A & B \\ * & * \end{smallmatrix}\right)$. Then $1 = \det(Y) = (-1)^{t-1}\det\left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right) = (-1)^t(b^2 - ad)$, where d is the $(2t,2t)$-entry of the matrix Y. The lemma is proved.

Let us now prove the "only if" parts of (b)–(d) of Theorem 3. We assume $(R_{t,n})$ and want to obtain certain conclusions about the ring R.

(b) Taking $(a,b) = (0,1)$ in Lemma 3, we conclude that $2 = 0$ in R, i.e. $2R = 0$. For every maximal ideal m of R and any b in R outside of m we can find an element r in R such that $rb - 1 \in m$. Applying Lemma 3 to $(a,b) = (rb-1, b)$, we obtain that $b^2 - 1 \in m$.

Therefore $\mathrm{card}(B/m) = 2$ for every maximal ideal m of R, hence $r^2 - r \in m$ for every m, i.e. $r^2 - r \in \mathrm{rad}(R)$ for every r in R, where $\mathrm{rad}(R)$ is the Jacobson radical of R, i.e. the intersection of all maximal ideals.

In particular, $1 + r^2 - r = 1 + r^2 + r \in \mathrm{GL}_1 R$ for every r in R. By Lemma 3 with $(a,b) = (0, 1+r+r^2)$, we obtain that $(1+r+r^2)^2 = 1 + r^2 + r^4 = 1$, hence $r^2 = r^4$ for all r in R.

(c) For any maximal ideal m of R and any b outside of m we have $rb - 1 \in m$ for some r in R. By Lemma 3 with $(a,b) = (rb-1, b)$, we obtain $b^2 \equiv 1 \pmod{m}$. So $\mathrm{card}(R/m) = 2$ or 3 for every maximal ideal m of R. By Lemma 3 with $(a,b) = (0, 1+r^3-r)$, we obtain that $(1+r^3-r)^2 = 1$ for all r in R. Setting here $r = 2$, we have $48 = 0$ in R. So the ring R splits into the direct product of its 2-primary component $3R$ and its 3-primary component $8R$.

In $3R$ we have $r^2 - r \in \mathrm{rad}(3R)$, so $(1 - r^2 + r)^2 = 1$, hence

$$(1 + r - r^2)^2 - 1 = (r^2 - r)(r^2 - r - 2) = (r^2 - 2r)(r^2 - 1) = 0$$

for all r in $3R$.

In the ring $8R$ we have $r^3 - r \in \mathrm{rad}(8R)$, so $(1 \pm (r^3 - r))^2 = 1$, hence $4(r^3 - r) = (r^3 - r) = 0$, so $(r-2)(r^3 - r) = (r^2 - 2r)(r^2 - 1) = 0$ for all r in $8R$.

Thus, $(r^2 - 2r)(r^2 - 1) = 0$ for all r in R.

(d) For an arbitrary unit u of R we pick some $X = X^T$ in $\mathrm{GL}_{2t-n}R$ with $\det(X) = u$ and define $A = \left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $M_{t,t}R$. We set $B = \left(\begin{smallmatrix} 0 \\ 1_{n-t} \end{smallmatrix}\right)$ in $M_{t,n-t}R$. If $\left(\begin{smallmatrix} A & B \\ * & * \end{smallmatrix}\right) = Y$ is a symmetric matrix in $\mathrm{SL}_n R$, then

$$1 = \det(Y) = (-1)^{n-t}\det(X) = (-1)^{n-t}u,$$

hence $u = (-1)^{n-t}$.

Thus, the group $\mathrm{GL}_1 R$ is trivial. In particular, $2 = 0$ in R.

Evidently, $(R_{t,n})$ implies $(R_{t-(2t-n),n-(2t-n)}) = (R_{n-t,2(n-t)})$. By (b) and (c) proved above, $r^4 - r^2 = 0$ when $n - t$ is odd and $(r^2 - 2r)(r^2 - 1) = 0$ when $n - t$ is even, for all r in R. Since $2R = 0$, $r^4 = r^2$ for all r in all cases.

Since $\mathrm{GL}_1 R$ is trivial, $\mathrm{rad}(R) = 0$. So $r^4 - r^2 = (r^2 - r)^2 = 0$ implies that $r^2 - r = 0$ for all r in R.

Now we have the "if" parts of (b)–(d) to prove. So we assume that the appropriate polynomials vanish on R, and we want to prove $(R_{t,n})$.

By Theorem 1, $\mathrm{sr}(R) \leq 1$. Let (A, B) be as in $(R_{t,n})$. If $t = n/2$ (cases (b) and (c)), then, by Lemma 1, $B - AC \in \mathrm{GL}_t R$ for some C in $M_{t,t} R$. So the matrix

$$\begin{aligned} Y &= \begin{pmatrix} 1 & 0 \\ C^T & 1 \end{pmatrix} \begin{pmatrix} A & B - AC \\ (B - AC)^T & 0 \end{pmatrix} \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} A & B \\ B^T & B^T C + C^T B - C^T AC \end{pmatrix} \end{aligned}$$

is symmetric with $\det(Y) = (-1)^t \det(B - AC)^2$, where 1 stands for 1_t.

In case (b), the identity $r^4 = r^2$ gives that $u^2 = 1$ for every unit u of R. The identity $2r = 0$ gives that $(-1)^t = -1 = 1$. So $\det(Y) = (-1)^t \det(B - AC)^2 = 1$.

In case (c) the identity $(r^2 - 2r)(r^2 - 1) = 0$ gives (when $r = 3$) that $24 = 0$ in R. So R splits into the direct product of a 2-primary component $3R$ and a 3-primary component $8R$. On $3R$, $u^2 - 2u$ is a unit for any unit u, so the identity gives tht $u^2 = 1$. On $8R$, setting $r = u - 1$, we obtain that $u^2(u - 1)(u - 2) = 0$, hence $(u - 1)(u - 2) = u^2 - 1 = 0$ for any unit u. Thus, $u^2 = 1$ on R for any unit u. In particular, $\det(Y) = \det(B - AC)^2 = 1$.

In case (d) our statement follows from Theroem 4 (which will be proved in the next section), because $\mathrm{GL}_1 R$ is trivial for any Boolean ring R.

PROOF OF THEOREM 4. First, we want to prove that $\mathrm{sr}(R) \leq 1$, assuming that the statement $(R_{t,n})$ with $\mathrm{SL}_n R$ replaced by $\mathrm{GL}_n R$ is true for some t, n such that $3/2 \leq n/2 \leq t < n$.

Let a, b be in R. We set

$$(A, B) = X_2 = \begin{pmatrix} a & 1 & 0 & 0 \\ 1 & b & 1 & 0 \end{pmatrix} \quad \text{when } t = 2 = n/2;$$

$$(A, B) = X_t = \begin{pmatrix} 0 & 0 & 1_{n-t-2} \\ 0 & X_2 & 0 \end{pmatrix} \quad \text{when } n = 2t > 4;$$

$$(A, B) = \begin{pmatrix} 1_{2t-n} & 0 \\ 0 & X_{n-t} \end{pmatrix} \quad \text{when } n > 2t \geq 4.$$

If $Y = \left(\begin{smallmatrix} A & B \\ * & * \end{smallmatrix}\right)$ is a symmetric matrix in $\mathrm{GL}_n R$, then

$$\mathrm{GL}_1 R \ni \det(Y) = (-1)^{n-t} \det \begin{pmatrix} a & 1 & 0 \\ 1 & b & 1 \\ 0 & 1 & y \end{pmatrix} = (-1)^{n-t}(y(ab - 1) - a),$$

where y is the (n, n)th entry of Y.

Thus, for any a, b in R, there is some y in R such that $y(ab-1)-a$ is a unit. Let us show that this implies that $\mathrm{sr}(R) \leq 1$ (as matter of fact, this is equivalent to $\mathrm{sr}(R) \leq 1$). Let (a_1, a_2) in $M_{1,2}R$ be unimodular. Then $a_1b_1 + a_2b_2 = 1$ for some b_1, b_2 in R. As above, we find y in R such that $y(a_1b_1 - 1) - a_1 \in \mathrm{GL}_1 R$. Since $a_1b_1 - 1 = -a_2b_2$, we have $ya_2b_2 + a_1 \in \mathrm{GL}_1 R$. So $a_1 + a_2c \in \mathrm{GL}_1 R$ for $c = yb_2$.

Assume now tht $\mathrm{sr}(R) \leq 1$. We have to prove $(R_{t,n})$ with $\mathrm{SL}_n R$ replaced by $\mathrm{GL}_n R$, where $t \leq n$. When $t \leq n/2$, we are done by Theorem 2. The case $t = n$ is trivial. So we assume that $n/2 \leq t \leq n-1$.

Let (A, B) be as in $(R_{t,n})$. If $t = n/2$, then by Lemma 1, $B - AC \in \mathrm{GL}_t R$ for some C in $M_{t,t}R$ and we are done by a computation in a previous section.

Let now $n/2 < t < n$. When we replace (t, n) by $(t+1, n+1)$, we make our statement (the modified $(R_{t,n})$ which we are proving) stronger. So. after a few such replacements, we can assume that n is divisible by $n-t$. Set $k = n/(n-t) - 1$.

Note that (A, B) can be completed to a symmetric matrix in $\mathrm{GL}_n R$ if and only if $(DAD^T, D(B+AC))$ can, where C is a matrix in $M_{t,t}R$ and $D \in \mathrm{GL}_t R$ (compare with the proof of Theorem 2). Since $\mathrm{sr}(R) \leq 1$, Lemma 1 and 2 allow us to find C, D satisfying $D(B+AC) = \left(\begin{smallmatrix} 0 \\ 1_{n-t} \end{smallmatrix}\right)$ in $M_{t,n-t}R$. So we can assume that $B = \left(\begin{smallmatrix} 0 \\ 1_{n-t} \end{smallmatrix}\right)$ from the beginning.

Repeating this argument, we can assume that the matrix (A, B) in $M_{t,n}R$ has $D_1, D_2, \ldots, D_{k+1}$ along the main diagonal with D_i in $M_{n-t,n-t}R$; 1_{n-t} on the line above the main diagonal and on the line below the diagonal; 0 elsewhere.

Applying Lemma 4 below with $s = n-t$ and $U_k = 1_s$ to the matrix $Y = \left(\begin{smallmatrix} A & B \\ B^T & X \end{smallmatrix}\right)$ in $M_{n,n}R$, we see that $\det(Y) = \det(V + WX)$ for some V, W in $M_{s,s}R$ with unimodular (V, W) and $VW^T = WV^T$. Using Lemma 5 below, we find a symmetric X such that $Y \in \mathrm{GL}_n R$.

Thus, Theorem 4 is reduced to the following two lemmas.

LEMMA 4. *Let s, k be natural numbers, R a ring, $D_1, \ldots, D_{k+1}, U_1, \ldots, U_k$ matrices in $M_{s,s}R$ such that $U_i = 1_s$ for $i < k$ and all D_i are symmetric. Let Y be the matrix in $M_{sk+s,sk+s}R$ with D_i along the main diagonal, 1_s along the line below the diagonal, U_i along the line above the diagonal, and 0 elsewhere. Then $\det(Y) = \det(VU_k + WD_{k+1})$ for some matrices V, W in $M_{s,s}R$ with unimodular (V, W) and symmetric VW^T.*

PROOF. We proceed by induction on k. When $k = 1$, $Y = \left(\begin{smallmatrix} D_1 & U_1 \\ 1_s & D_2 \end{smallmatrix}\right)$, so

$$\det(Y) = \det\left(\begin{pmatrix} 1_s & -D_1 \\ 0 & 1_s \end{pmatrix} Y\right) = \det\begin{pmatrix} 0 & U_1 - D_1D_2 \\ 1_s & D_2 \end{pmatrix} = \det(D_1D_2 - U_1),$$

so we can take $V = D_1$ and $W = -1_s$.

Suppose now that $k > 1$. Using 1_s in the last s rows of $Y = F(D_1, \ldots, D_{k+1}; U_k)$, we can eliminate the term D_{k+1} by column addition operations over R. Then it is clear that $\det(Y) = \det(F(D_1, \ldots, D_{k-1}, D_kD_{k-1} - U_k; D_{k+1}))$.

By the induction hypothesis, the last determinant has the form

$$\det(V'D_{k+1} + W'(D_kD_{k+1} - U_k))$$

with unimodular (V', W') and symmetric VW^T. But $V'D_{k+1}+W'(D_kD_{k+1}-U_k) = VU_k + WD_{k+1}$ with

$$(V,W) = (-W', V' + W'D_k) = (-W', V')\begin{pmatrix} 1_s & -D_k \\ 0 & 1_s \end{pmatrix}$$

which is unimodular, because so is (V', W'). Moreover, $WV^T = -(V'+W'D_k)W'^T$ is symmetric, because so are D_k and $V'W'^T$.

LEMMA 5. *Let R be a ring with* $\mathrm{sr}(R) \leq 1$. *Then for any natural number s and any matrices V and W in $M_{s,s}R$ such that (V,W) in $M_{s,2s}R$ is unimodular and $VW^T = WV^T$ there is a matrix $X = X^T$ in $M_{s,s}R$ such that $V + WX \in \mathrm{GL}_s R$.*

PROOF. We proceed by induction on s. When $s = 1$, our conclusion coincides with our condition $\mathrm{sr}(R) \leq 1$. Let now $s \geq 2$.

By Lemma 1 and 2, there are matrices C in $M_{s,s}R$ and D in $\mathrm{GL}_s R$ such that $D(V + WC) = 1_s$. Let X_1 be the symmetric matrix in $M_{s,s}R$ with the same first column as C and with all entries outside the first column and row equal 0. Then $D(V + WV_1) = \left(\begin{smallmatrix} 1 & u \\ 0 & V' \end{smallmatrix}\right)$ with some u in $M_{1,s-1}R$ and V' in $M_{s-1,s-1}R$. We write $DW = \left(\begin{smallmatrix} * & w \\ v & W' \end{smallmatrix}\right)$ with v, w^T in $M_{s-1,1}R$ and W' in $M_{s-1,s-1}R$.

Since (V,W) is unimodular, so is $D(U + WX_1, W) = \left(\begin{smallmatrix} 1 & * & * & * \\ 0 & V' & v & W' \end{smallmatrix}\right)$, hence the matrix (V', v, W') in $M_{s-1,2s-1}R$ is also unimodular. Since WV^T is symmetric, so is

$$DW(D(V + WX_1))^T = \begin{pmatrix} * & w \\ v & W' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u^T & V'^T \end{pmatrix} = \begin{pmatrix} * & wV'^T \\ v + W'u^T & W'V'^T \end{pmatrix},$$

hence $v + W'u^T = V'w^T$ and $W'V'^T = V'W'^T$. Since (V', v, W') is unimodular and $v = V'w^T - W'u^T$, we conclude that (V', W') is unimodular.

By the induction hypothesis, $V' + W'X' \in \mathrm{GL}_{s-1}R$ for some symmetric matrix X' in $M_{s-1,s-1}R$. We set $X = X_1 + \left(\begin{smallmatrix} 0 & 0 \\ 0 & X' \end{smallmatrix}\right) = X^T$ in $M_{s,s}R$. Then

$$\begin{aligned} D(V + WX) &= D(V + WX_1) + DW\begin{pmatrix} 0 & 0 \\ 0 & X' \end{pmatrix} \\ &= \begin{pmatrix} 1^* & \\ 0 & V' \end{pmatrix} + \left(\begin{pmatrix} * & * \\ * & W' \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & X' \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 & * \\ 0 & V' + W'X' \end{pmatrix} \in \mathrm{GL}_s R, \end{aligned}$$

hence $V + WX \in \mathrm{GL}_s R$.

REFERENCES

1. H. Bass, *K-theory and stable algebra*, Inst. Hautes Études Sci. Publ. Math. **22** (1964), 485–544.
2. M. Newman, *Matrix completion theorems*, Proc. Amer. Math. Soc. **94** (1985), 39–45.
3. L. N. Vaserstein, *The stable range of rings and dimension of topological spaces*, Funktsional. Anal. i Prilozhen. **5** (1971), 17–27 (translated in Functional Anal. Appl.).
4. ——, *Stabilization for unitary and orthogonal groups over a ring with involution*, Mat. Sb. **81** (1970), 328–352 (translated in Math. USSR-Sb. **10**).

5. ——, *Stabilization for the classical groups over rings*, Mat. Sb. **93** (1974), 268–295 (translated in Math. USSR-Sb. **22**).
6. ——, *Bass's first stable range conditions*, J. Pure Appl. Algebra **34** (1984), 319–330.
7. L. N. Vaserstein and A. A. Suslin, *Serre's problem on projective modules over polynomial rings and algebraic K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), 993–1054 (translated in Math. USSR-Izv. **10**).

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540 (Current address)