AN ANSWER TO A QUESTION OF M. NEWMAN
ON MATRIX COMPLETION
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ABSTRACT. Let $R$ be a principal ideal ring, $A$ a symmetric $t$-by-$t$ matrix over $R$, $B$ a $t$-by-$(n - t)$ matrix over $R$ such that the $t$-by-$n$ matrix $(A, B)$ is primitive. Newman [2] proved that $(A, B)$ may be completed (as the first $t$ rows) to a symmetric $n$-by-$n$ matrix of determinant 1, provided that $1 < t < n/3$. He showed that the result is false, in general, if $t = n/2$, and he asked to determine all values of $t$ such that $1 < t < n$ and the result holds. It is shown here that these values are exactly $t$ satisfying $1 < t < n/2$.

Moreover, the result is proved for a larger (than the principal ideal rings) class of commutative rings, namely, for the rings satisfying the second stable range condition of Bass [1].

Also, it is observed that Theorems 2 and 3 of [2, p. 40] proved there for principal ideal rings are true for this larger class of rings, as well as the basic result of [2, p. 39].

Statement of results. In this paper, “ring” means “commutative associative ring with 1”.

We use the notation $sr(R)$ [3, 7]. In particular, $sr(R) \leq 1$ means that the following condition (the first stable range condition of Bass [1]) holds:

$(sr(R) \leq 1)$. If $a_1, a_2$ are in $R$ and $a_1 R + a_2 R = R$, then $(a_1 + a_2 c) R = R$ for some $c$ in $R$.

This condition was discussed in detail in [6], where many examples are given. A. Roy showed me principal ideal rings with $sr(R) \leq 1$ which are not semilocal. B. Magurn showed me Dedekind rings $R$ with $sr(R) \leq 1$ which are not principal ideal rings. The following theorem gives more examples (including all Boolean rings $R$ with 1).

THEOREM 1. Suppose that there is a primitive polynomial $f(x)$ in one variable $x$ with integral coefficients such that $f(r) = 0$ for every element $r$ of a ring $R$. Then $sr(R) \leq 1$.

The second stable range condition, $sr(R) \leq 2$, reads as follows:

$(sr(R) \leq 2)$. If $a_1, a_2, a_3$ are in $R$ and $a_1 R + a_2 R + a_3 R = R$, then $(a_1 + a_3 c_1) R + (a_2 + a_3 c_2) R = R$ for some $c_1, c_2$ in $R$.

If $R$ is Noetherian of finite Krull dimension $dim(R)$, then $sr(R) \leq 1 + dim(R)$, by Bass [1]. In particular, $sr(R) \leq 1 + dim(R) \leq 1 + 1 = 2$ for each principal ideal ring $R$.

Here are more examples of rings $R$ with $sr(R) \leq 2$: The ring of integers in any number field or, more generally, any subring $R$ with 1 of any global field; any $R$ whose space of maximal ideals is Noetherian of dimension $\leq 1$; any finitely generated ideal...
generated algebra $R$ over any finite field with $\dim(R) \leq 2$ (see [7]); the ring of real continuous functions on an interval or, more generally, on any topological space of dimension $\leq 1$ (see [3]); the ring of all continuous complex functions on a topological space of dimension $\leq 3$ (see [3])

For any ring $R$ and any natural numbers $t$ and $n$, let $M_{t,n}R$ denote the set of all $t$-by-$n$ matrices over $R$. A matrix in $M_{t,n}R$ is said to be unimodular if it has a left (when $t \leq n$) or right (when $t > n$) inverse in $M_{n,t}R$. This is equivalent for this matrix to be primitive, i.e. the ideal of $R$ generated by all maximal minors to be $R$.

Let $GL_{n}R$ be the group of unimodular (i.e. invertible) matrices in $M_{n,n}R$, and let $SL_{n}R$ denote the subgroup of matrices with determinant 1.

**Theorem 2.** Let $R$ be a ring, and let $t$ and $n$ be natural numbers such that $t < n$. If $sr(R) < n - 2t + 1$ and $t < n/2$, then

$(R_{t,n})$. For every symmetric matrix $A = A^T$ in $M_{t,t}$ and every matrix $B$ in $M_{t,n-t}R$ such that the matrix $(A, B)$ in $M_{t,n}R$ is unimodular, there is a symmetric matrix in $SL_{n}R$ whose first $t$ rows form the given matrix $(A, B)$.

**Theorem 3.** Let $R$ be a ring, and let $t$ and $n$ be natural numbers such that $t \leq n$. Then

(a) when $t = n$, the statement $(R_{t,n})$ above is true if and only if the group $GL_{1}R$ is trivial, i.e. 1 is the only unit of $R$;

(b) when $t = n/2$ is odd, $(R_{t,n})$ is true if and only if $r^4 = r^2$ for all $r$ in $R$ and $2R = 0$;

(c) when $t = n/2$ is even, $(R_{t,n})$ is true if and only if $(r^2 - 2r)(r^2 - 1) = 0$ for all $r$ in $R$;

(d) when $n/2 < t < n$, $(R_{t,n})$ is true if and only if $r^2 = r$ for all $r$ in $R$, i.e. $R$ is a Boolean ring.

Combining Theorems 2 and 3 we obtain the following complete answer to the open question (c) of [2, p. 45].

**Corollary.** If $sr(R) \leq 2$ (for example, $R$ is a principal ideal ring) then the statement $(R_{t,s})$ holds, provided that $1 \leq t < n/2$. If $(r - 2)(r^4 - r^2) \neq 0$ for some $r$ in $R$, then the statement $(R_{t,n})$ never holds for $n/2 < t < n$.

However, Theorem 3 justifies a modification of the question of Newman for $t \geq n/2$. For example, $(R_{n,n})$ becomes true if we replace $SL_{n}R$ in it by $GL_{n}R$. The following theorem answers the modified question for all $t$.

**Theorem 4.** The statement $(R_{t,n})$ with $SL_{n}R$ replacea by $GL_{n}R$ is true for all $t$ and $n$ such that $t \leq n$ if and only if $sr(R) \leq 1$.

**Remark.** Let $sr(R) \leq 2$. Then, by [4, 5], for any natural numbers $t$ and $n$ such that $t \leq n$ and any matrix $C$ in $M_{t,t}R$, the symplectic group

$$Sp_{2n}R = \left\{ D \in SL_{2n}R : D^T \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} D = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\}$$

acts transitively on the matrices $(A, B)$ in $M_{t,2n}R$ with $A, B$ in $M_{t,n}R$ and $AB^T = BA^T = C$. 

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Taking here $C = 0$ and $t = 1$ (respectively, $t = n$), we obtain Theorem 2 (resp., Theorem 3) of [2], extended from principal ideal rings $R$ to any $R$ with $sr(R) \leq 2$.

**Proof of Theorem 1.** Let $d$ be the degree of the polynomial $f(x)$. Then for any maximal ideal $m$ of $R$ the polynomial vanishes on the field $R/m$, hence $\text{card}(R/m) \leq d$. Therefore $r^d \equiv r \pmod{m}$ for every element $r$ in $R$ and any maximal ideal $m$ of $R$.

Let $(a_1, a_2)$ be an arbitrary unimodular row in $M_{1,2}R$. We set $c = a_1^d - 1$. The claim is that $(a_1 + a_2c)R = R$. It suffices to show that the left-hand side is not contained in any maximal ideal $m$ of $R$. Passing to the factor ring $R/m$, we see that this suffices to prove the claim in the case when $R$ is a field. In this case,

$$a_1 + a_2c = \begin{cases} a_1 & \text{if } a \neq 0 \text{ (because then } c = 0), \\ a_2 & \text{if } a_1 = 0 \text{ (because then } c = -1) \end{cases}$$

So $a_1 + a_2c \neq 0$, hence $(a_1 + a_2c)R = R$.

**Proof of Theorem 2.** We will use the following two lemmas:

**Lemma 1 (see [3]).** For any natural numbers $d, s, t$ the condition $sr(R) \leq d$ is equivalent to the following condition: For any matrices $A$ in $M_{s,d+s-1}R$ and $B$ in $M_{s,t}R$ such that $(A, B)$ in $M_{s,d+s-1+t}R$ is unimodular there is a matrix $C$ in $M_{t,r+d+s-1}R$ such that the matrix $A + BC$ in $M_{t,r+d+s-1}R$ is unimodular.

**Lemma 2 (see [1, 5, 7]).** For any $s, t$ with $t \geq s + sr(R)$, the group $SL_tR$ acts transitively on the set of unimodular matrices in $M_{s,t}R$; if $sr(R) \leq 2$, then the condition $t \geq s + sr(R)$ above can be replaced by $t \geq s + 1$.

**Remark.** Lemma 2 can be easily proved by induction on $s$. After this, it is not hard to prove Lemma 1 also by induction on $s$. Of course, Lemma 2 is a trivial consequence of Lemma 1. Lemma 2 generalizes the basic result of [2, p. 39].

Now we are ready to prove Theorem 2. Let $sr(R) \leq n - 2t + 1$ and let $A$ and $B$ be as in $(R_{t,n})$.

**Case 1.** $B$ has the form $(1_t, 0)$, where $1_t$ is the identity matrix in $SL_tR$ and $0$ is the zero matrix in $M_{t,n-2t}R$. Then for any matrix $X = X^T$ in $M_{n-2t,n-2t}R$ the matrix

$$Y = \begin{pmatrix} A & 1_t & 0 \\ 1_t & 0 & 0 \\ 0 & 0 & X \end{pmatrix} \quad \text{in } M_{n,n}R$$

is symmetric, its first $t$ rows form the given matrix $(A, B) = (A, 1_t, 0)$, and $\det(Y) = (-1)^t \det(X)$. Choosing $X$ with $\det(X) = (-1)^t$ (a diagonal matrix will do), we complete our proof in Case 1.

**Case 2.** $B$ is unimodular. Then by Lemma 2 there is a matrix $C$ in $SL_{n-t}R$ such that $BC = (1_t, 0)$. By Case 1, there is a symmetric matrix $Y$ in $SL_nR$ of the form $Y = (A_{1,BC})$. Then the symmetric matrix

$$\begin{pmatrix} 1_t & 0 \\ 0 & C^{-1} \end{pmatrix}^T Y \begin{pmatrix} 1_t & 0 \\ 0 & C^{-1} \end{pmatrix} \quad \text{in } SL_nR$$

is of the form $(A_{1,B})$.
General case. By Lemma 1, there is a matrix $D$ in $M_{t,n-t}R$ such that $B + AD$ in $M_{t,n-t}R$ is unimodular. By Case 2, there is a symmetric matrix $Y$ in $SL_nR$ of the form $Y = \begin{pmatrix} A & B + AD \end{pmatrix}$. Then the symmetric matrix

$$\begin{pmatrix} 1_t & 0 \\ -D^T & 1_{n-t} \end{pmatrix} Y \begin{pmatrix} 1_t & -D \\ 0 & 1_{n-t} \end{pmatrix}$$

is of the form $\begin{pmatrix} A & B \end{pmatrix}$.

**Proof of Theorem 3.** Part (a) is evident, so let $n/2 < t < n$.

**Lemma 3.** The condition $(R_t,2t)$ implies that $b^2 \equiv (-1)^t \pmod{aR}$ for every unimodular row $(a,b)$ in $M_{t,t}R$.

**Proof.** We set $A = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ and $B = \begin{pmatrix} 1_{t-1} & 0 \\ 0 & b \end{pmatrix}$ (both in $M_{t,t}R$; when $t = 1$, $(A,B) = (a,b)$). Let $Y$ be a symmetric matrix in $SL_{2t}R$ of the form $Y = \begin{pmatrix} A & B \end{pmatrix}$. Then $1 = \det(Y) = (-1)^{t-1} \det(\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}) = (-1)^t(b^2 - ad)$, where $d$ is the $(2t,2t)$-entry of the matrix $Y$. The lemma is proved.

Let us now prove the “only if” parts of (b)–(d) of Theorem 3. We assume $(R_t,n)$ and want to obtain certain conclusions about the ring $R$.

(b) Taking $(a,b) = (0,1)$ in Lemma 3, we conclude that $2 = 0$ in $R$, i.e. $2R = 0$. For every maximal ideal $m$ of $R$ and any $b$ in $R$ outside of $m$ we can find an element $r$ in $R$ such that $rb - 1 \in m$. Applying Lemma 3 to $(a,b) = (rb - 1, b)$, we obtain that $b^2 - 1 \in m$.

Therefore $\text{card}(R/m) = 2$ for every maximal ideal $m$ of $R$, hence $r^2 - r \in m$ for every $m$, i.e. $r^2 - r \in \text{rad}(R)$ for every $r$ in $R$, where $\text{rad}(R)$ is the Jacobson radical of $R$, i.e. the intersection of all maximal ideals.

In particular, $1 + r^2 - r = 1 + r^2 + r \in GL_1R$ for every $r$ in $R$. By Lemma 3 with $(a,b) = (0,1 + r + r^2)$, we obtain that $(1 + r + r^2)^2 = 1 + r^2 + r^4 = 1$, hence $r^2 = r^4$ for all $r$ in $R$.

(c) For any maximal ideal $m$ of $R$ and any $b$ outside of $m$ we have $rb - 1 \in m$ for some $r$ in $R$. By Lemma 3 with $(a,b) = (rb - 1, b)$, we obtain $b^2 \equiv 1 \pmod{m}$.

So $\text{card}(R/m) = 2$ or $3$ for every maximal ideal $m$ of $R$. By Lemma 3 with $(a,b) = (0,1 + r^3 - r)$, we obtain that $(1 + r^3 - r)^2 = 1$ for all $r$ in $R$. Setting here $r = 2$, we have $48 = 0$ in $R$. So the ring $R$ splits into the direct product of its 2-primary component $3R$ and its 3-primary component $8R$.

In $3R$ we have $r^2 - r \in \text{rad}(3R)$, so $(1 - r^2 + r)^2 = 1$, hence

$$(1 + r^2 - r^2)^2 - 1 = (r^2 - r)(r^2 - r - 2) = (r^2 - 2r)(r^2 - 1) = 0$$

for all $r$ in $3R$.

In the ring $8R$ we have $r^3 - r \in \text{rad}(8R)$, so $(1 \pm (r^3 - r))^2 = 1$, hence $4(r^3 - r) = (r^3 - r) = 0$, so $(r - 2)(r^3 - r) = (r^2 - 2r)(r^2 - 1) = 0$ for all $r$ in $8R$.

Thus, $(r^2 - 2r)(r^2 - 1) = 0$ for all $r$ in $R$.

(d) For an arbitrary unit $u$ of $R$ we pick some $X = X^T$ in $GL_{2t-n}R$ with $\det(X) = u$ and define $A = (\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix})$ in $M_{t,t}R$. We set $B = (\begin{pmatrix} 0 \\ 1_{n-t} \end{pmatrix})$ in $M_{t,n-t}R$. If $(A,B) = Y$ is a symmetric matrix in $SL_nR$, then

$$1 = \det(Y) = (-1)^{n-t} \det(X) = (-1)^{n-t} u,$$

hence $u = (-1)^{n-t}$. 

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Thus, the group $\text{GL}_1 R$ is trivial. In particular, $2 = 0$ in $R$.

Evidently, $(R_{t,n})$ implies $(R_{t-(2t-n),n-(2t-n)}) = (R_{n-t,2(n-t)})$. By (b) and (c) proved above, $r^4 - r^2 = 0$ when $n - t$ is odd and $(r^2 - 2r)(r^2 - 1) = 0$ when $n - t$ is even, for all $r$ in $R$. Since $2R = 0$, $r^4 = r^2$ for all $r$ in all cases.

Since $\text{GL}_1 R$ is trivial, $\text{rad}(R) = 0$. So $r^4 - r^2 = (r^2 - r)^2 = 0$ implies that $r^2 - r = 0$ for all $r$ in $R$.

Now we have the “if” parts of (b)-(d) to prove. So we assume that the appropriate polynomials vanish on $R$, and we want to prove $(R_{t,n})$.

By Theorem 1, $sr(R) \leq 1$. Let $(A, B)$ be as in $(R_{t,n})$. If $t = n/2$ (cases (b) and (c)), then, by Lemma 1, $B - AC \in \text{GL}_1 R$ for some $C$ in $M_{t,t} R$. So the matrix

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C^T & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is symmetric with $\det(Y) = (-1)^t \det(B - AC)^2$, where $1$ stands for $1_t$.

In case (b), the identity $r^4 = r^2$ gives that $u^2 = 1$ for every unit $u$ of $R$. The identity $2r = 0$ gives that $(-1)^t = -1 = 1$. So $\det(Y) = (-1)^t \det(B - AC)^2 = 1$.

In case (c) the identity $(r^2 - 2r)(r^2 - 1) = 0$ gives (when $r = 3$) that $24 = 0$ in $R$. So $R$ splits into the direct product of a 2-primary component $3R$ and a 3-primary component $8R$. On $3R$, $u^2 - 2u$ is a unit for any unit $u$, so the identity gives that $u^2 = 1$. On $8R$, setting $r = u - 1$, we obtain that $u^2(u - 1)(u - 2) = 0$, hence $(u - 1)(u - 2) = u^2 - 1 = 0$ for any unit $u$. Thus, $u^2 = 1$ on $R$ for any unit $u$. In particular, $\det(Y) = \det(B - AC)^2 = 1$.

In case (d) our statement follows from Theorem 4 (which will be proved in the next section), because $\text{GL}_1 R$ is trivial for any Boolean ring $R$.

**Proof of Theorem 4.** First, we want to prove that $sr(R) \leq 1$, assuming that the statement $(R_{t,n})$ with $\text{SL}_n R$ replaced by $\text{GL}_n R$ is true for some $t, n$ such that $3/2 \leq n/2 \leq t < n$.

Let $a, b$ be in $R$. We set

$$(A, B) = X_2 = \begin{pmatrix} a & 1 & 0 & 0 \\ 1 & b & 1 & 0 \end{pmatrix} \quad \text{when } t = 2 = n/2;$$

$$(A, B) = X_t = \begin{pmatrix} 0 & 0 & 1_{n-t-2} \\ 0 & X_2 & 0 \end{pmatrix} \quad \text{when } n = 2t > 4;$$

$$(A, B) = \begin{pmatrix} 1_{2t-n} & 0 \\ 0 & X_{n-t} \end{pmatrix} \quad \text{when } n > 2t \geq 4.$$
Thus, for any \( a, b \) in \( R \), there is some \( y \) in \( R \) such that \( y(ab - 1) - a \) is a unit. Let us show that this implies that \( \text{sr}(R) \leq 1 \) (as matter of fact, this is equivalent to \( \text{sr}(R) \leq 1 \)). Let \((a_1, a_2)\) in \( M_{1,2}R \) be unimodular. Then \( a_1b_1 + a_2b_2 = 1 \) for some \( b_1, b_2 \) in \( R \). As above, we find \( y \) in \( R \) such that \( y(a_1b_1 - 1) - a_1 \in \text{GL}_1R \). Since \( a_1b_1 - 1 = -a_2b_2 \), we have \( ya_2b_2 + a_1 \in \text{GL}_1R \). So \( a_1 + a_2c \in \text{GL}_1R \) for \( c = yb_2 \).

Assume now that \( \text{sr}(R) \leq 1 \). We have to prove \((R_t, n)\) with \( \text{SL}_nR \) replaced by \( \text{GL}_nR \), where \( t \leq n \). When \( t \leq n/2 \), we are done by Theorem 2. The case \( t = n \) is trivial. So we assume that \( n/2 \leq t < n \).

Let \((A, B)\) be as in \((R_t, n)\). If \( t = n/2 \), then by Lemma 1, \( B - AC \in \text{GL}_tR \) for some \( C \) in \( M_{t, t}R \) and we are done by a computation in a previous section.

Let now \( n/2 < t < n \). When we replace \((t, n)\) by \((t + 1, n + 1)\), we make our statement (the modified \((R_t, n)\) which we are proving) stronger. So, after a few such replacements, we can assume that \( n \) is divisible by \( n - t \). Set \( k = n/(n - t) - 1 \).

Note that \((A, B)\) can be completed to a symmetric matrix in \( \text{GL}_nR \) if and only if \((DAD^T, D(B + AC))\) can, where \( C \) is a matrix in \( M_{t, t}R \) and \( D \in \text{GL}_tR \) (compare with the proof of Theorem 2). Since \( \text{sr}(R) \leq 1 \), Lemma 1 and 2 allow us to find \( C, D \) satisfying \( D(B + AC) = \begin{pmatrix} 0 \\ 1_{n-t} \end{pmatrix} \) in \( M_{t, n-t}R \). So we can assume that \( B = \begin{pmatrix} \alpha \\ 1_{n-t} \end{pmatrix} \) from the beginning.

Repeating this argument, we can assume that the matrix \((A, B)\) in \( M_{t, n}R \) has \( D_1, D_2, \ldots, D_{k+1} \) along the main diagonal with \( D_i \) in \( M_{n-t, n-t}R \); \( 1_{n-t} \) on the line above the main diagonal and on the line below the diagonal; 0 elsewhere.

Applying Lemma 4 below with \( s = n - t \) and \( U_k = 1_{s} \) to the matrix \( Y = \begin{pmatrix} A & B \\ B^T & X \end{pmatrix} \) in \( M_{n, n}R \), we see that \( \det(Y) = \det(V + WX) \) for some \( V, W \) in \( M_{s, s}R \) with unimodular \((V, W)\) and \( V^TW = WV^T \). Using Lemma 5 below, we find a symmetric \( X \) such that \( Y \in \text{GL}_nR \).

Thus, Theorem 4 is reduced to the following two lemmas.

**Lemma 4.** Let \( s, k \) be natural numbers, \( R \) a ring, \( D_1, \ldots, D_{k+1}, U_1, \ldots, U_k \) matrices in \( M_{s, s}R \) such that \( U_i = 1_s \) for \( i < k \) and all \( D_i \) are symmetric. Let \( Y \) be the matrix in \( M_{sk+s, sk+s}R \) with \( D_i \) along the main diagonal, \( 1_s \) along the line below the diagonal, \( U_i \) along the line above the diagonal, and 0 elsewhere. Then \( \det(Y) = \det(VU_k + WD_{k+1}) \) for some matrices \( V, W \) in \( M_{s, s}R \) with unimodular \((V, W)\) and symmetric \( V^TW \).

**Proof.** We proceed by induction on \( k \). When \( k = 1 \), \( Y = \begin{pmatrix} D_1 & U_1 \\ U_1 & D_2 \end{pmatrix} \), so

\[
\det(Y) = \det\left( \begin{pmatrix} 1_s & -D_1 \\ 0 & 1_s \end{pmatrix} \right) = \det\left( \begin{pmatrix} 0 \\ 1_s \end{pmatrix} U_1 - D_1D_2 \right) = \det(D_1D_2 - U_1),
\]

so we can take \( V = D_1 \) and \( W = -1_s \).

Suppose now that \( k > 1 \). Using \( 1_s \) in the last \( s \) rows of \( Y = F(D_1, \ldots, D_{k+1}; U_k) \), we can eliminate the term \( D_{k+1} \) by column addition operations over \( R \). Then it is clear that \( \det(Y) = \det(F(D_1, \ldots, D_{k-1}, D_kD_{k-1} - U_k; D_{k+1})) \).

By the induction hypothesis, the last determinant has the form

\[
\det(V'D_{k+1} + W'(D_kD_{k+1} - U_k))
\]
with unimodular \((V', W')\) and symmetric \(VW^T\). But \(V'D_{k+1} + W'(D_kD_{k+1} - U_k) = VU_k + WD_{k+1}\) with

\[(V, W) = (-W', (V' + W'D_k)) = (-W', V') \begin{pmatrix} 1_s & -D_k \\ 0 & 1_s \end{pmatrix},\]

which is unimodular, because so is \((V', W')\). Moreover, \(WV^T = -(V' + W'D_k)W'^T\) is symmetric, because so are \(D_k\) and \(V'W'T\).

**Lemma 5.** Let \(R\) be a ring with \(sr(R) \leq 1\). Then for any natural number \(s\) and any matrices \(V\) and \(W\) in \(M_{s,s}R\) such that \((V, W)\) in \(M_{s,2s}R\) is unimodular and \(VW^T = WV^T\) there is a matrix \(X = XT\) in \(M_{s,s}SR\) such that \(V + WX \in GL_sR\).

**Proof.** We proceed by induction on \(s\). When \(s = 1\), our conclusion coincides with our condition \(sr(R) \leq 1\). Let now \(s \geq 2\).

By Lemma 1 and 2, there are matrices \(C\) in \(M_{s,s}R\) and \(D\) in \(GL_sR\) such that \(D(V + WC) = 1_s\). Let \(X_1\) be the symmetric matrix in \(M_{s,s}R\) with the same first column as \(C\) and with all entries outside the first column and row equal 0. Then \(D(V + WV_1) = (1_u V')\) with some \(u\) in \(M_{s-1,s-1}R\) and \(V'\) in \(M_{s-1,s-1}R\). We write \(DW = (V' W') \) with \(v, w^T\) in \(M_{s-1,s-1}R\) and \(W'\) in \(M_{s-1,s-1}R\).

Since \((V, W)\) is unimodular, so is \(D(U + WX_1, W) = (1_{V'} V' W')\), hence the matrix \((V', v, W')\) in \(M_{s-1,2s-1}R\) is also unimodular. Since \(WV^T\) is symmetric, so is \(V + WX\) unimodular, so is \(D(U + WX_1, W) = (1_{V'} V' W')\), hence the matrix \((V', v, W')\) in \(M_{s-1,2s-1}R\) is also unimodular. Since \(WV^T\) is symmetric, so is

\[ DW(D(V + WX_1))^T = \begin{pmatrix} * & w \\ v & W' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u^T & V'^T \end{pmatrix} = \begin{pmatrix} * & WV^T \\ v + W'u^T & W'V^T \end{pmatrix}, \]

hence \(v + W'u^T = V'w^T\) and \(V'V'^T = V'W'V^T\). Since \((V', v, W')\) is unimodular and \(v = V'w^T - W'u^T\), we conclude that \((V', W')\) is unimodular.

By the induction hypothesis, \(V' + W'X' \in GL_{s-1}R\) for some symmetric matrix \(X'\) in \(M_{s-1,s-1}R\). We set \(X = X_1 + (0, 0, \ldots, X') = XT\) in \(M_{s,s}R\). Then

\[ D(V + WX) = D(V + WX_1) + DW(\begin{pmatrix} 0 & 0 \\ 0 & X' \end{pmatrix}) = \begin{pmatrix} 1 & * \\ 0 & V' \end{pmatrix} + \begin{pmatrix} * & * \\ * & W' \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & X' \end{pmatrix}, \]

hence \(V + WX \in GL_sR\).

**References**


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