THE nTH ROOTS OF SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we shall prove the following theorem: Let $K$ be a differential field of characteristic zero. Let $\phi$ and $\psi$ be elements of a differential field extension of $K$ such that (i) $\phi \neq 0$ and $\psi \neq 0$; (ii) $\phi$ and $\psi$ satisfy nontrivial linear differential equations with coefficients in $K$, say, $P(\phi) = 0$ and $Q(\psi) = 0$; (iii) $\phi = \psi^n$ for some positive integer $n$ such that $n \geq \text{ord } P$. Then the logarithmic derivatives of $\phi$ and $\psi$ are algebraic over $K$. (Note that $\phi'/\phi = n(\psi'/\psi)$.)

1. Introduction. Let $K$ be a differential field of characteristic zero. We denote by $\mathcal{D}_K$ the ring of linear ordinary differential operators with coefficients in $K$, that is

$$\mathcal{D}_K = \left\{ \sum_{k=0}^{m} a_k D^k; a_k \in K, m \in N \right\},$$

where $D$ denotes differentiation in any differential field extension of $K$ and $N$ is the set of all nonnegative integers. In a previous paper [2] we characterized those functions which together with their reciprocal satisfy linear differential equations:

**THEOREM A.** Let $\phi$ be an element of a differential field extension of $K$ such that

(i) $\phi \neq 0$;
(ii) $P(\phi) = 0$ for some $P \in \mathcal{D}_K - \{0\}$;
(iii) $Q(1/\phi) = 0$ for some $Q \in \mathcal{D}_K - \{0\}$.

Then, the logarithmic derivative of $\phi$ ($= \phi'/\phi$) is algebraic over $K$.

In the present paper, utilizing a similar method, we shall prove the following result:

**THEOREM B.** Let $\phi$ and $\psi$ be elements of a differential field extension of $K$ such that

(i) $\phi \neq 0$ and $\psi \neq 0$;
(ii) $P(\phi) = 0$ for some $P \in \mathcal{D}_K - \{0\}$;
(iii) $Q(\psi) = 0$ for some $Q \in \mathcal{D}_K - \{0\}$;
(iv) $\phi = \psi^n$ for some positive integer $n$ such that

$$n \geq \text{ord } P.$$

Then $\phi'/\phi = n(\psi'/\psi)$ is algebraic over $K$. 
Theorem B gives the following two corollaries:

**COROLLARY C.** In case $K$ is the field of rational functions of $x$ with coefficients in $C$, if $\varphi$ and $\psi$ satisfy the hypotheses of Theorem B, then $\varphi'/\varphi = n(\psi'/\psi)$ is an algebraic function of $x$.

**COROLLARY D.** In case $K$ is the quotient field of $C\{x\}$ (the ring of convergent power series in $x$ with coefficients in $C$), if $\varphi$ and $\psi$ are formal power series in $x$ with coefficients in $C$, and if $\varphi$ and $\psi$ satisfy the hypotheses of Theorem B, then $\varphi$ and $\psi \in C\{x\}$ (i.e., $\varphi$ and $\psi$ are convergent).

The following examples illustrate these results.

**EXAMPLE 1.** The power series $\{\cos x\}_{1/n}^{1/n}$ and $J_\nu(x)^{1/n} (n > 2)$ do not satisfy any linear ordinary differential equation with polynomial coefficient (cf. also [1]).

**EXAMPLE 2.** The divergent power series $\{\sum_{m=0}^\infty m!x^m\}_{1/n}^{1/n} (n > 2)$ do not satisfy any linear ordinary differential equation with coefficients in $C\{x\}$.

2. **A fundamental lemma.** The following two important steps were utilized in our previous paper [2].

(1) Set

\[
P = \sum_{h=0}^{m+1} p_h D^h, \quad Q = \sum_{h=0}^{n+1} q_h D^h,
\]

where $m$ and $n \in N$; $p_h$ and $q_h \in K$; in particular,

\[
p_{m+1} \neq 0, \quad q_{n+1} \neq 0.
\]

For an element of $f$ of $K$, let us set

\[
\hat{f} = \sum_{h=0}^{\infty} \frac{f^{(h)}}{h!} x^h \in K[[x]],
\]

where $K[[x]]$ is the ring of formal power series in $x$ with coefficients in $K$. Then, $T(f) = \hat{f}$ defines an injective homomorphism of rings $T: K \to K[[x]]$ such that $T(f') = dT(f)/dx$. Corresponding to two operators $P$ and $Q$ of (2.1), let us consider two operators

\[
\hat{P} = \sum_{h=0}^{m+1} \hat{p}_h (d/dx)^h, \quad \hat{Q} = \sum_{h=0}^{n+1} \hat{q}_h (d/dx)^h.
\]

We assume that $\varphi$ is an element of a differential field extension of $K$. Denote this extension by $\tilde{K}$. Then, $P(\varphi) = 0$ and $Q(1/\varphi) = 0$ imply, respectively, that the formal power series

\[
\hat{\varphi} = \sum_{h=0}^{\infty} \frac{\varphi^{(h)}}{h!} x^h \in \tilde{K}[[x]]
\]

satisfies $\hat{P}(\hat{\varphi}) = 0$ and $\hat{Q}(1/\hat{\varphi}) = 0$.

Observe that (2.2) implies $1/\hat{p}_{m+1} \in K[[x]]$ and $1/\hat{q}_{n+1} \in K[[x]]$. Therefore

\[
y = \frac{\hat{\varphi}}{\varphi} = 1 + \sum_{h=1}^{\infty} \frac{\varphi^{(h)}}{\varphi h!} x^h.
\]
satisfies the differential equation

\[(2.4)\quad y^{(m+1)} + \sum_{h=0}^{m} \hat{p}_h y^{(h)} = 0,\]

and \(u = \varphi/\hat{\varphi}\) satisfies the differential equation

\[(2.5)\quad u^{(n+1)} + \sum_{h=0}^{n} \hat{q}_h u^{(h)} = 0.\]

In this manner, the general case was reduced to the case of formal power series.

(2) Let \(k\) be a field. We denote by \(k[y_1, \ldots, y_p]\) the ring of polynomials in \(p\) indeterminates \(y_1, \ldots, y_p\) with coefficients in \(k\). For \(F \in k[y_1, \ldots, y_p]\) we set

\[(2.6)\quad w(F) = \deg_t (\alpha_1 t, \alpha_2 t^2, \ldots, \alpha_p t^p),\]

where we regard \(F(\alpha_1 t, \alpha_2 t^2, \ldots, \alpha_p t^p)\) as a polynomial in \(t\) whose coefficients are polynomials in \(\alpha_1, \ldots, \alpha_p\). The following lemma was the fundamental algebraic tool in the proof of Theorem A.

**Lemma E.** Let \(n\) and \(p\) be positive integers, and let \(F_1, \ldots, F_p \in k[y_1, \ldots, y_p]\). Assume that

(i) \(w(F_p - y_p^n) < np\);

(ii) \(w(F_j) = nj\) (\(j = 1, \ldots, p - 1\));

(iii) \(w(F_j(y_1, \ldots, y_p, 0, \ldots, 0) - y_j^n) < nj\) (\(j = 1, \ldots, p - 1\)).

Then, the system of algebraic equations

\[(2.7)\quad F_j(y_1, \ldots, y_p) = 0 \quad (j = 1, \ldots, p)\]

admits only a finite number of solutions in any field extension of \(k\) and these solutions are algebraic over \(k\).

3. A lemma on differential equations. Let \(k\) be a field of characteristic zero, and let \(k[[x]]\) be the ring of formal power series in \(x\) with coefficients in \(k\). Then a linear ordinary differential equation

\[(3.1)\quad y^{(m+1)} + \sum_{h=0}^{m} a_h(x)y^{(h)} = 0 \quad (a_h \in k[[x]])\]

admits a canonical basis of \(m + 1\) solutions of the form

\[(3.2)\quad f_j = \frac{1}{j!} x^j + \sum_{h=m+1}^{\infty} f_{j,h} x^h \quad (j = 0, 1, \ldots, m),\]

where \(f_{j,h} \in k\).

Let \(k\) be a field extension of \(k\). If a formal power series

\[(3.3)\quad \varphi = \sum_{h=0}^{m} c_h x^h\]

with coefficients in \(\tilde{k}\) satisfies (3.1), then we have

\[(3.4)\quad \varphi = \sum_{j=0}^{m} (j!) c_j f_j.\]
Let us consider another linear ordinary differential equation:

\[(3.5) \quad u^{(p+1)} + \sum_{h=0}^{p} b_h(x) u^{(h)} = 0 \quad (b_h \in k[[x]]).\]

This equation also has a canonical basis of \(p + 1\) solutions of the form

\[(3.6) \quad g_j = \frac{1}{j!} x^j + \sum_{h=p+1}^{\infty} g_{j,h} x^h \quad (j = 0, 1, \ldots, p),\]

where \(g_{j,h} \in k\).

As in our previous paper [2], in order to prove Theorem B it suffices to prove the following result:

**Lemma F.** There are at most a finite number of solutions of the form

\[(3.7) \quad \begin{cases} y = f_0 + \sum_{j=1}^{m} \alpha_j f_j, & (\alpha_j \in \tilde{k}), \\ u = g_0 + \sum_{j=1}^{p} \beta_j g_j, & (\beta_j \in \tilde{k}) \end{cases}\]

of (3.1) and (3.5) respectively such that

\[(3.8) \quad y = u^n\]

if

\[(3.9) \quad n \geq m + 1.\]

For these solutions, the coefficients \(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_p\) are algebraic over \(k\).

**Proof.** Set \(y = \sum_{h=0}^{\infty} \lambda_h x^h\) and \(u = \sum_{h=0}^{\infty} \mu_h x^h\), where \(\lambda_h\) and \(\mu_h \in \tilde{k}\). Then

\[(3.10) \quad \begin{cases} \lambda_0 = 1, & \mu_0 = 1, \\ \lambda_j = (1/j!) \alpha_j & (j = 1, \ldots, m), \\ \mu_j = (1/j!) \beta_j & (j = 1, \ldots, p), \\ \lambda_h = f_{0,h} + \sum_{j=1}^{m} (j!) \alpha_j f_{j,h} & (h \geq m + 1), \\ \mu_h = g_{0,h} + \sum_{j=1}^{p} (j!) \beta_j g_{j,h} & (h \geq p + 1). \end{cases}\]

In order that

\[(3.11) \quad \sum_{h=0}^{\infty} \lambda_h x^h = \left( \sum_{h=0}^{\infty} \mu_h x^h \right)^n,\]

it is necessary and sufficient that

\[(3.12) \quad \lambda_h = \sum_{h_1 + \cdots + h_n = h, h_j \geq 0} \mu_{h_1} \cdots \mu_{h_n} \quad \text{for } h \geq 0.\]

Let us define \(w(F)\) for \(F \in k[\mu_1, \ldots, \mu_p]\) in the same way as (2.6), and let us make the following observations:

1. Utilizing the last formulas of (3.10), we regard \(\mu_h \quad (h \geq p + 1)\) as elements of \(k[\mu_1, \ldots, \mu_p]\). Then

\[(3.13) \quad w(\mu_h) \leq p < h \quad \text{for } h \geq p + 1.\]

2. Utilizing \(\lambda_h = \sum \mu_{h_1} \cdots \mu_{h_n} \quad \text{for } h = 1, \ldots, m\) (cf. (3.12)), we can regard \(\lambda_1, \ldots, \lambda_m\) as elements of \(k[\mu_1, \ldots, \mu_p]\). Then

\[(3.14) \quad w(\lambda_j) = j \quad (j = 1, \ldots, m).\]
(3) Utilizing the formulas for $\lambda_h$ ($h \geq m + 1$) of (3.10), we can regard $\lambda_h$ ($h \geq m + 1$) as elements of $k[\mu_1, \ldots, \mu_p]$. Then

\begin{equation}
(3.15) \quad w(\lambda_h) \leq m < h \quad \text{for } h \geq m + 1.
\end{equation}

Now, regarding (3.12) for $h \geq m + 1$ as a system of algebraic equations on $\mu_1, \ldots, \mu_p$, let us consider the system of $p$ equations

\begin{equation}
(3.16) \quad \lambda_h = \sum_{h_1 + \cdots + h_n = h, h_j \geq 0} \mu_{h_1} \cdots \mu_{h_n} = 0
\end{equation}

where $h = n, 2n, \ldots, pn$. Set

\begin{equation}
(3.17) \quad F_j(\mu_1, \ldots, \mu_p) = \lambda_{jn} - \sum_{h_1 + \cdots + h_n = n_j} \mu_{h_1} \cdots \mu_{h_n} \quad (j = 1, \ldots, p).
\end{equation}

Note that, if (3.9) is satisfied, we have

\begin{equation}
(3.18) \quad w(\lambda_{jn}) < n_j \quad (j = 1, \ldots, p)
\end{equation}

(cf. (3.15)). Therefore,

\begin{equation}
(3.19) \quad w(F_j) \leq n_j \quad (j = 1, \ldots, p)
\end{equation}

if (3.9) is satisfied. Furthermore, since

\begin{equation}
F_j(\mu_1, \ldots, \mu_j, 0, \ldots, 0) = \lambda_{jn} - \sum \mu_{h_1} \cdots \mu_{h_n}
\end{equation}

if $\mu_{j+1} = 0, \ldots, \mu_p = 0$, where the sum is over all $n$-tuples $(h_1, \ldots, h_n)$ such that $h_1 + \cdots + h_n = n_j$, $h_j \geq 0$, it follows that

\begin{equation}
(3.20) \quad w(F_j(\mu_1, \ldots, \mu_j, 0, \ldots, 0) - \mu_j^n) < n_j \quad (j = 1, \ldots, p).
\end{equation}

Thus applying Lemma E to (3.16) we can complete the proof of Lemma F, and thus Theorem B.

Using different methods, our results have been substantially generalized in [3].

REFERENCES


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