PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 2, June 1986

# THE $n$TH ROOTS OF SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS[1]

WILLIAM A. HARRIS, JR. AND YASUTAKA SIBUYA[2]

ABSTRACT. In this paper we shall prove the following theorem: Let $K$ be a differential field of characteristic zero. Let $\varphi$ and $\psi$ be elements of a differential field extension of $K$ such that (i) $\varphi \neq 0$ and $\psi \neq 0$; (ii) $\varphi$ and $\psi$ satisfy nontrivial linear differential equations with coefficients in $K$, say, $P(\varphi) = 0$ and $Q(\psi) = 0$; (iii) $\varphi = \psi^n$ for some positive integer $n$ such that $n \geq \operatorname{ord} P$. Then the logarithmic derivatives of $\varphi$ and $\psi$ are algebraic over $K$. (Note that $\varphi'/\varphi = n(\psi'/\psi)$.)

**1. Introduction.** Let $K$ be a differential field of characteristic zero. We denote by $\mathcal{D}_K$ the ring of linear ordinary differential operators with coefficients in $K$, that is

$$\mathcal{D}_K = \left\{ \sum_{k=0}^{m} a_k D^k ; a_k \in K, m \in N \right\},$$

where $D$ denotes differentiation in any differential field extension of $K$ and $N$ is the set of all nonnegative integers. In a previous paper [**2**] we characterized those functions which together with their reciprocal satisfy linear differential equations:

THEOREM A. *Let $\varphi$ be an element of a differential field extension of $K$ such that*

(i) $\varphi \neq 0$;
(ii) $P(\varphi) = 0$ *for some* $P \in \mathcal{D}_K - \{0\}$;
(iii) $Q(1/\varphi) = 0$ *for some* $Q \in \mathcal{D}_K - \{0\}$.

*Then, the logarithmic derivative of $\varphi$ $(= \varphi'/\varphi)$ is algebraic over $K$.*

In the present paper, utilizing a similar method, we shall prove the following result:

THEOREM B. *Let $\varphi$ and $\psi$ be elements of a differential field extension of $K$ such that*

(i) $\varphi \neq 0$ *and* $\psi \neq 0$;
(ii) $P(\varphi) = 0$ *for some* $P \in \mathcal{D}_K - \{0\}$;
(iii) $Q(\psi) = 0$ *for some* $Q \in \mathcal{D}_K - \{0\}$;
(iv) $\varphi = \psi^n$ *for some postive integer $n$ such that*

$$n \geq \operatorname{ord} P. \tag{1.1}$$

*Then $\varphi'/\varphi = n(\psi'/\psi)$ is algebraic over $K$.*

Received by the editors June 24, 1985.
1980 *Mathematics Subject Classification*. Primary 12H05, 13B20, 13J05, 34A30.

[1]This paper was presented in a special session on ordinary differential equations at the 91st annual meeting of the AMS on January 10, 1985 at Anaheim, California.

[2]Both authors were supported in part by grants from the National Science Foundation.

©1986 American Mathematical Society
0002-9939/86 $1.00 + $.25 per page

Theorem B gives the following two corollaries:

COROLLARY C. *In case $K$ is the field of rational functions of $x$ with coefficients in* $\mathbf{C}$, *if $\varphi$ and $\psi$ satisfy the hypotheses of Theorem* B, *then $\varphi'/\varphi = n(\psi'/\psi)$ is an algebraic function of $x$.*

COROLLARY D. *In case $K$ is the quotient field of $\mathbf{C}\{x\}$ (the ring of convergent power series in $x$ with coefficients in* $\mathbf{C}$), *if $\varphi$ and $\psi$ are formal power series in $x$ with coefficients in* $\mathbf{C}$, *and if $\varphi$ and $\psi$ satisfy the hypotheses of Theorem* B, *then $\varphi$ and $\psi \in \mathbf{C}\{x\}$ (i.e., $\varphi$ and $\psi$ are convergent).*

The following examples illustrate these results.

EXAMPLE 1. The power series $\{\cos x\}^{1/n}$ and $J_\nu(x)^{1/n}$ $(n \geq 2)$ do not satisfy any linear ordinary differential equation with polynomial coefficient (cf. also [**1**]).

EXAMPLE 2. The divergent power series $\{\sum_{m=0}^{\infty} m! x^m\}^{1/n}$ $(n \geq 2)$ do not satisfy any linear ordinary differential equation with coefficients in $\mathbf{C}\{x\}$.

**2. A fundamental lemma.** The following two important steps were utilized in our previous paper [**2**].

(1) Set

$$P = \sum_{h=0}^{m+1} p_h D^h, \qquad Q = \sum_{h=0}^{n+1} q_h D^h, \tag{2.1}$$

where $m$ and $n \in N$; $p_h$ and $q_h \in K$; in particular,

$$p_{m+1} \neq 0, \qquad q_{n+1} \neq 0. \tag{2.2}$$

For an element of $f$ of $K$, let us set

$$\hat{f} = \sum_{h=0}^{\infty} \frac{f^{(h)}}{h!} x^h \in K[[x]],$$

where $K[[x]]$ is the ring of formal power series in $x$ with coefficients in $K$. Then, $T(f) = \hat{f}$ defines an injective homomorphism of rings $T\colon K \to K[[x]]$ such that $T(f') = dT(f)/dx$. Corresponding to two operators $P$ and $Q$ of (2.1), let us consider two operators

$$\hat{P} = \sum_{h=0}^{m+1} \hat{p}_h (d/dx)^h, \qquad \hat{Q} = \sum_{h=0}^{n+1} \hat{q}_h (d/dx)^h.$$

We assume that $\varphi$ is an element of a differential field extension of $K$. Denote this extension by $\tilde{K}$. Then, $P(\varphi) = 0$ and $Q(1/\varphi) = 0$ imply, respectively, that the formal power series

$$\hat{\varphi} = \sum_{h=0}^{\infty} \frac{\varphi^{(h)}}{h!} x^h \in \tilde{K}[[x]]$$

satisfies $\hat{P}(\hat{\varphi}) = 0$ and $\hat{Q}(1/\hat{\varphi}) = 0$.

Observe that (2.2) implies $1/\hat{p}_{m+1} \in K[[x]]$ and $1/\hat{q}_{n+1} \in K[[x]]$. Therefore

$$y = \hat{\varphi}/\varphi = 1 + \sum_{h=1}^{\infty} \frac{\varphi^{(h)}}{\varphi h!} x^h \tag{2.3}$$

satisfies the differential equation

$$y^{(m+1)} + \sum_{h=0}^{m} \frac{\hat{p}_h}{\hat{p}_{m+1}} y^{(h)} = 0, \tag{2.4}$$

and $u = \varphi/\hat{\varphi}$ satisfies the differential equation

$$u^{(n+1)} + \sum_{h=0}^{n} \frac{\hat{q}_h}{\hat{q}_{n+1}} u^{(h)} = 0. \tag{2.5}$$

In this manner, the general case was reduced to the case of formal power series.

(2) Let $k$ be a field. We denote by $k[y_1, \ldots, y_p]$ the ring of polynomials in $p$ indeterminates $y_1, \ldots, y_p$ with coefficients in $k$. For $F \in k[y_1, \ldots, y_p]$ we set

$$w(F) = \deg_t F(\alpha_1 t, \alpha_2 t^2, \ldots, \alpha_p t^p), \tag{2.6}$$

where we regard $F(\alpha_1 t, \alpha_2 t^2, \ldots, \alpha_p t^p)$ as a polynomial in $t$ whose coefficients are polynomials in $\alpha_1, \ldots, \alpha_p$. The following lemma was the fundamental algebraic tool in the proof of Theorem A.

LEMMA E. *Let $n$ and $p$ be positive integers, and let $F_1, \ldots, F_p \in k[y_1, \ldots, y_p]$. Assume that*

(i) $w(F_p - y_p^n) < np$;

(ii) $w(F_j) = nj$ $(j = 1, \ldots, p-1)$;

(iii) $w(F_j(y_1, \ldots, y_j, 0, \ldots, 0) - y_j^n) < nj$ $(j = 1, \ldots, p-1)$.

*Then, the system of algebraic equations*

$$F_j(y_1, \ldots, y_p) = 0 \qquad (j = 1, \ldots, p) \tag{2.7}$$

*admits only a finite number of solutions in any field extension of $k$ and these solutions are algebraic over $k$.*

**3. A lemma on differential equations.** Let $k$ be a field of characteristic zero, and let $k[[x]]$ be the ring of formal power series in $x$ with coefficients in $k$. Then a linear ordinary differential equation

$$y^{(m+1)} + \sum_{h=0}^{m} a_h(x) y^{(h)} = 0 \qquad (a_h \in k[[x]]) \tag{3.1}$$

admits a canonical basis of $m+1$ solutions of the form

$$f_j = \frac{1}{j!} x^j + \sum_{h=m+1}^{\infty} f_{j,h} x^h \qquad (j = 0, 1, \ldots, m), \tag{3.2}$$

where $f_{j,h} \in k$.

Let $\tilde{k}$ be a field extension of $k$. If a formal power series

$$\varphi = \sum_{h=0}^{m} c_h x^h \tag{3.3}$$

with coefficients in $\tilde{k}$ satisfies (3.1), then we have

$$\varphi = \sum_{j=0}^{m} (j!) c_j f_j. \tag{3.4}$$

Let us consider another linear ordinary differential equation:

$$u^{(p+1)} + \sum_{h=0}^{p} b_h(x) u^{(h)} = 0 \qquad (b_h \in k[[x]]). \tag{3.5}$$

This equation also has a canonical basis of $p+1$ solutions of the form

$$g_j = \frac{1}{j!} x^j + \sum_{h=p+1}^{\infty} g_{j,h} x^h \qquad (j = 0, 1, \ldots, p), \tag{3.6}$$

where $g_{j,h} \in k$.

As in our previous paper [**2**], in order to prove Theorem B it suffices to prove the following result:

LEMMA F. *There are at most a finite number of solutions of the form*

$$\begin{cases} y = f_0 + \sum_{j=1}^{m} \alpha_j f_j & (\alpha_j \in \tilde{k}), \\ u = g_0 + \sum_{j=1}^{p} \beta_j g_j & (\beta_j \in \tilde{k}) \end{cases} \tag{3.7}$$

*of* (3.1) *and* (3.5) *respectively such that*

$$y = u^n \tag{3.8}$$

*if*

$$n \geq m + 1. \tag{3.9}$$

*For these solutions, the coefficients* $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_p$ *are algebraic over* $k$.

PROOF. Set $y = \sum_{h=0}^{\infty} \lambda_h x^h$ and $u = \sum_{h=0}^{\infty} \mu_h x^h$, where $\lambda_h$ and $\mu_h \in \tilde{k}$. Then

$$\begin{cases} \lambda_0 = 1, \qquad \mu_0 = 1, \\ \lambda_j = (1/j!)\alpha_j \qquad (j = 1, \ldots, m), \\ \mu_j = (1/j!)\beta_j \qquad (j = 1, \ldots, p), \\ \lambda_h = f_{0,h} + \sum_{j=1}^{m} (j!) f_{j,h} \lambda_j \qquad (h \geq m+1), \\ \mu_h = g_{0,h} + \sum_{j=1}^{p} (j!) g_{j,h} \mu_j \qquad (h \geq p+1). \end{cases} \tag{3.10}$$

In order that

$$\sum_{h=0}^{\infty} \lambda_h x^h = \left( \sum_{h=0}^{\infty} \mu_h x^h \right)^n, \tag{3.11}$$

it is necessary and sufficient that

$$\lambda_h = \sum_{h_1 + \cdots + h_n = h, h_j \geq 0} \mu_{h_1} \cdots \mu_{h_n} \quad \text{for } h \geq 0. \tag{3.12}$$

Let us define $w(F)$ for $F \in k[\mu_1, \ldots, \mu_p]$ in the same way as (2.6), and let us make the following observations:

(1) Utilizing the last formulas of (3.10), we regard $\mu_h$ $(h \geq p+1)$ as elements of $k[\mu_1, \ldots, \mu_p]$. Then

$$w(\mu_h) \leq p < h \quad \text{for } h \geq p+1. \tag{3.13}$$

(2) Utilizing $\lambda_h = \sum \mu_{h_1} \cdots \mu_{h_n}$ for $h = 1, \ldots, m$ (cf. (3.12)), we can regard $\lambda_1, \ldots, \lambda_m$ as elements of $k[\mu_1, \ldots, \mu_p]$. Then

$$w(\lambda_j) = j \qquad (j = 1, \ldots, m). \tag{3.14}$$

(3) Utilizing the formulas for $\lambda_h$ $(h \geq m+1)$ of (3.10), we can regard $\lambda_h$ $(h \geq m+1)$ as elements of $k[\mu_1, \ldots, \mu_p]$. Then

$$w(\lambda_h) \leq m < h \quad \text{for } h \geq m+1. \tag{3.15}$$

Now, regarding (3.12) for $h \geq m+1$ as a system of algebraic equations on $\mu_1, \ldots, \mu_p$, let us consider the system of $p$ equations

$$\lambda_h - \sum_{h_1+\cdots+h_n=h, h_j \geq 0} \mu_{h_1} \cdots \mu_{h_n} = 0 \tag{3.16}$$

where $h = n, 2n, \ldots, pn$. Set

$$F_j(\mu_1, \ldots, \mu_p) = \lambda_{jn} - \sum_{h_1+\cdots+h_n=nj} \mu_{h_1} \cdots \mu_{h_n} \qquad (j = 1, \ldots, p). \tag{3.17}$$

Note that, if (3.9) is satisfied, we have

$$w(\lambda_{jn}) < nj \qquad (j = 1, \ldots, p) \tag{3.18}$$

(cf. (3.15)). Therefore,

$$w(F_j) \leq nj \qquad (j = 1, \ldots, p) \tag{3.19}$$

if (3.9) is satisfied. Furthermore, since

$$F_j(\mu_1, \ldots, \mu_j, 0, \ldots, 0) = \lambda_{jn} - \sum \mu_{h_1} \cdots \mu_{h_n}$$

if $\mu_{j+1} = 0, \ldots, \mu_p = 0$, where the sum is over all $n$-tuples $(h_1, \ldots, h_n)$ such that $h_1 + \cdots + h_n = nj$, $h_j \geq 0$, it follows that

$$w(F_j(\mu_1, \ldots, \mu_j, 0, \ldots, 0) - \mu_j^n) < nj \qquad (j = 1, \ldots, p). \tag{3.20}$$

Thus applying Lemma E to (3.16) we can complete the proof of Lemma F, and thus Theorem B.

Using different methods, our results have been substantially generalized in [**3**].

## References


1. R. P. Stanley, *Differentiably finite power series*, European J. Combin. **1** (1980), 175–188.
2. W. A. Harris, Jr. and Y. Sibuya, *The reciprocals of solutions of linear ordinary differential equations*, IMA Preprint Series No. 72, Univ. of Minnesota, May 1984; Adv. in Math. (to appear).
3. S. Sperber, *On solutions of differential equations which satisfy certain algebraic relations*, Pacific J. of Math. (to appear).



Department of Mathematics, University of Southern California, Los Angeles, California 90089-1113

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455