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A GROUP THEORETICAL EQUIVALENT OF THE ZERO DIVISOR PROBLEM

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ABSTRACT. We prove that the zero divisor problem is equivalent to some problem about subgroups of a free group.

1. Introduction. Let G be an arbitrary torsion free group. The problem of whether the group ring KG contains zero divisor is open, although positive solution has been obtained for some important special classes of groups, in particular, for polycyclic-by-finite-groups (Farkas and Snider [**1**], Cliff [**2**]) and for one-relator groups (J. Lewin and T. Lewin [**3**]). In this note we show that in the cases when K is the ring of integers Z or the field Z_p of residue classes modulo p the zero divisor problem for KG is equivalent to a problem about subgroups of a free group.

Let $G = F/N$ be a presentation of G as a quotient group of a free group F; as usual $\gamma_k(N)$ $(k = 1, 2, \ldots)$ denotes the kth term of the lower central series of N. If $1 \neq x \in N$ then there exists a maximal number k such that $x \in \gamma_k(N)$; we can say too that $x \in \gamma_k(N) \backslash \gamma_{k+1}(N)$.

THEOREM 1. *The ring ZG contains no zero divisors if for every system of elements $x, f_1, f_2, \ldots, f_n \in F$, where $x \in \gamma_k(N)\backslash\gamma_{k+1}(N)$ and is not a power in $\gamma_k(N)$ (and hence not in N) and f_i $(i = 1, 2, \ldots, n)$ are taken from different cosets of G/N, the subgroup N_1, generated by the elements*

$$x_i = f_i^{-1} x f_i \qquad (i = 1, 2, \ldots, n) \tag{1}$$

satisfies the condition

$$N_1 \cap \gamma_{k+1}(N) = N_1'. \tag{2}$$

The inverse assertion is true without the restriction that x is not a power in $\gamma_k(N)\backslash\gamma_{k+1}(N)$.

PROPOSITION 1. *Assume that ZG contains no zero divisors. Then for every element $x \in \gamma_k(N)\backslash\gamma_{k+1}(N)$ and $f_1, f_2, \ldots, f_n$ taken from different cosets of F/N, the elements* (1) *are free generators of the subgroup N_1 and* (2) *holds.*

REMARK. Since $x \in \gamma_k(N)$ we have $N_1 \subseteq \gamma_k(N)$ and thus $N_1' \subseteq \gamma_{k+1}(N)$, and hence

$$N_1 \cap \gamma_{k+1}(N) \supseteq N_1'.$$

Thus, the real sense of condition (2) is that $N_1 \cap \gamma_{k+1}(N)$ does not properly include N_1'. □

When the element x in Theorem 1 does not belong to N', i.e., $k = 1$, we immediately obtain the following corollary.

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COROLLARY. *The ring ZG contains no zero divisors iff*

$$N_1 \cap N' = N_1', \tag{3}$$

for all choices of $x, f_1, f_2, \ldots, f_n$ as described.

It is worth remarking that our group theoretical equivalent of the zero divisor problem is obtained for group rings over Z or Z_p; the connection between the zero divisor problem over Z or Z_p and over an arbitrary field K is unknown.

2. Let F be a free group, $N \triangleleft F$ and $G = F/N$. It is easy to see that the free abelian group N/N' becomes a ZG-module via the conjugation in F; this is, the so-called relation module which has been studied intensively during the recent years. We refer the reader to K. Gruenberg's lecture notes [**4**] for the detailed definition and properties of these modules.

For $k > 1$ the factors $\gamma_k(N)/\gamma_{k+1}(N)$ become ZG-modules in the same way. We need the following property of these higher relation modules:

LEMMA 1. *For every natural k the ZG-module $\gamma_k(N)/\gamma_{k+1}(N)$ can be embedded in a free module.*

PROOF. The result is known and is proven even for a more general type of module: the higher relative relation module (see [**5**]). We give, therefore, only the outline of the proof.

When $k = 1$, the embedding of the relation module N/N' into a free module M is described in [**4**, lecture 2]. It is based on the existence of the exact sequence of ZG-modules

$$0 \to N/N' \to M \to \omega(ZG) \to 0, \tag{4}$$

where M is a free ZG-module and $\omega(ZG)$ is the augmentation ideal of ZG. When $k > 1$ the assertion follows from the observation that $\gamma_k(N)/\gamma_{k+1}(N)$ is embedded into the kth tensor power

$$M^{(k)} = \underbrace{M \otimes_Z M \otimes \cdots \otimes_Z M}_{k}.$$

Clearly, $M^{(k)}$ is a free ZG-module.

LEMMA 2. *Let R be a prime ring, and let M be a left nonzero R-module which can be embedded into a free module. Then R contains no zero divisors iff for any elements $r \in R$, $m \in M$, the condition*

$$rm = 0 \tag{5}$$

implies necessarily that $r = 0$ or $m = 0$.

PROOF. Let $0 \neq r \in R$. If (5) holds, then r must annihilate all the coordinates of m and hence, in the case when R contains no zero divisors, we obtain that $m = 0$. Conversely, let x, y be two nonzero elements in R such that $xy = 0$.

Consider the subset $yM \subseteq M$. Clearly, $x(yM) = 0$ and it remains to prove that $yM \neq 0$. But if $yM = 0$ we obtain a nonzero two sided ideal $A = (y)$ such that

$$AM = 0. \tag{6}$$

Now let M be embedded into a free R-module with a basis e_i $(i \in I)$. The ith coordinates of elements from M form a left ideal $B_i \subseteq R$ $(i \in I)$ and we obtain from (6) that

$$AB_i = 0 \qquad (i \in I). \tag{7}$$

Since $M \neq 0$ at least one of these left ideals B_i is not zero and we conclude easily via the assumption that R is prime that (7) is impossible, i.e., $yM \neq 0$ and the assertion follows.

3. Proofs of Theorem 1 and Proposition 1. Since G is torsion free, ZG must be prime (see [**6**]). Lemma 2 implies that ZG contains zero divisors iff there exist nonzero

$$r = \sum_{i=1}^{n} \alpha_i g_i \in ZG \quad \text{and} \quad \bar{x} \in M = \gamma_k(N)/\gamma_{k+1}(N)$$

such that

$$\sum_{i=1}^{n} \alpha_i (g_i \bar{x}) = 0, \tag{8}$$

or, in other words, ZG contains zero divisors iff there exist elements $g_i \in G$ $(i = 1, \dots, n)$ and $0 \neq \bar{x} \in M$ such that elements $g_i\bar{x}$ are linearly dependent over Z. Pick an arbitrary coset representative f_i in the different cosets $\bar{f_i} = f_i N$ $(i = 1, 2, \dots, n)$ and x in $\bar{x} = x\gamma_{k+1}(N)$. We can formulate the last condition:

(∗) *The elements $f_i^{-1} x f_i$ $(i = 1, 2, \dots, n)$ of $\gamma_k(N)$ are linearly dependent modulo $\gamma_{k+1}(N)$.*

We now prove Proposition 1. Since ZG is a domain the condition (∗) implies that the elements $f_i^{-1} x f_i$ $(i = 1, 2, \dots, n)$, which generate N_1, are linearly independent modulo $N_1 \cap \gamma_{k+1}(N)$ and hence are linearly independent modulo N_1'. We therefore conclude that these elements freely generate N_1. It is easy to see now that their linear independence modulo $N_1 \cap \gamma_{k+1}(N) \supseteq N_1'$ implies (2) and the proof of Proposition 1 is completed.

We prove now Theorem 1. If ZG contains zero divisors then (∗) holds *for some $x \in \gamma_k(N)$. Furthermore, we can assume that x is not a power in $\gamma_k(N)$.* Indeed, this is obvious if $\bar{x}$ is not a power in the free abelian group $\overline{M} = \gamma_k(N)/\gamma_{k+1}(N)$; but if there exists $\bar{y} \in M$ such that $x = n\bar{y}$ then we can replace $\bar{x}$ by $\bar{y}$ in (8) and then replace x by y in (∗).

We now observe that the quotient group $F/\gamma_k(N)$ is torsion free: when $k = 1$ it coincides with the torsion free group G; when $k > 1$ this fact follows for instance from [**7**]. We can therefore conclude that x is not a power in F.

Now let T be the normal subgroup of F, generated by the element x. Clearly, $T \subseteq N$; hence the elements f_i $(i = 1, 2, \dots, n)$ belong to different cosets of G/T. The quotient group G/T is a torsion-free one-relator group and we obtain from Lyndon's Identity Theorem (see [**8**]) that the elements (1) are linearly independent in T/T'. But the subgroup N_1, generated by these elements, belongs to T; this implies easily that the elements (1) are linearly independent in $N_1/N_1 \cap T'$ and hence in N_1/N_1'. Once again, condition (∗) implies that zero divisors in ZG exist iff the elements (1) are linearly dependent in $N_1/N_1 \cap \gamma_{k+1}(N)$, i.e., iff $N_1 \cap \gamma_{k+1}(N)$ properly contains N_1' and the assertion follows.

4. To obtain an analog of Theorem 1 for the ring Z_pG we have to consider instead of the lower central series the p-series $M_j(N)$ $(j = 1, 2, \ldots)$ of the dimension subgroups of Z_pN (see [**6**, Chapter 11]). We remind the reader that $M_1(N) = N$, $M_2(N) = N'N^p$ (the product of the commutator subgroup and the subgroup N^p, generated by the pth powers of all the element of N), that all the factors $M_j(N)/M_{j+1}(N)$ are abelian groups of exponent p, and that

$$\bigcap_{j=1}^{\infty} M_j(N) = 1. \tag{9}$$

Once again, the quotient group $M_j(N)/M_{j+1}(N)$ becomes a Z_pG-module via the conjugation in F.

LEMMA 3. *For every given j the Z_pG-module $M_j(N)/M_{j+1}(N)$ can be embedded into a free module.*

PROOF. The module $M_1(N)/M_2(N) = N/N'N^p$ is isomorphic to the Z_pG-module $\overline{N}/(p\overline{N})$, where $\overline{N}$ is the relation module N/N'. This implies via (4) the existence of the exact sequence of Z_pG-modules

$$0 \to M_1(N)/M_2(N) \to M/pM \to \omega(Z_pG) \to 0. \tag{4'}$$

Once again, $M_j(N)/M_{j+1}(N)$ is embedded into the jth tensor power of the free Z_pG-module M/pM. □

THEOREM 2. *The ring Z_pG contains no zero divisors iff for every system of elements $x, f_1, f_2, \ldots, f_n \in F$, where $x \in M_j(N) \backslash M_{j+1}(N)$ and is not a power in N and f_i $(i = 1, 2, \ldots, n)$ are taken from different cosets of G/N, the subgroup N_1, generated by the elements* (1), *satisfies the condition*

$$N_1 \cap M_{j+1}(N) = M_2(N_1). \tag{2'}$$

REMARK 1. It is worth remarking that the inclusion $N_1 \cap M_{j+1}(N) \supseteq M_2(N_1)$ follows immediately from the condition $x \in M_j(N)$.

REMARK 2. If ZG contains no zero divisors then (2′) holds without the assumption that x is not a power in N. The proof of this fact is similar to the proof of Proposition 1 and we omit it.

LEMMA 4. *Let H be a free group, and let U be a normal subgroup of finite index. Then every coset hU $(h \in H)$ contains an element which is not a proper power in H.*

PROOF. Let h_i $(i \in I)$ be a free system of generators for H and let

$$h = h_{i_1}^{\alpha_1} h_{i_2}^{\alpha_2} \cdots h_{i_k}^{\alpha_k};$$

pick n such that $h_{i_1}^n \in H$ and $\alpha_1 + n > \alpha_i$ $(i = 1, 2, \ldots, n-1)$. It is easy to verify that the element

$$h_{i_1}^{(n+\alpha_1)} h_{i_2}^{\alpha_2} \cdots h_{i_k}^{\alpha_k} = h_{i_1}^n h$$

is not a power in H.

PROOF OF THEOREM 2. We only give the outline of the proof since the argument is essentially the same as in Theorem 1.

First, find $0 \neq \bar{x} \in M_j(N)/M_{j+1}(N)$ such that (8) holds (with coefficients $\alpha_i \in Z_p$). Then find x in the coset $\bar{x}$ such that x is not a proper power in N, and hence in F. Then, similarly to condition $(*)$, obtain that the elements (1) of $M_j(N)$ are linearly dependent modulo $M_{j+1}(N)$.

On the other hand, the elements (1) are linearly independent in $T/T'T^p$ and hence in $N_1/N_1'N^p$; thus, there exist zero divisors in Z_pG iff $N_1 \cap M_{j+1}(N)$ properly contains $M_2(N_1)$.

COROLLARY. *Let $x \in N \backslash N'N^p$ and $f_1, f_2, \ldots, f_k$ be as in Theorem 2. Then Z_pG contains no zero divisors iff $N_1 \cap N'N^p = N_1'N_1^p$.*

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