ON THE CLIFFORD INDEX OF ALGEBRAIC CURVES
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ABSTRACT. Here we prove (over $\mathbb{C}$) that a general $(e+2)$-gonal algebraic curve of genus $p$ has no $g^r_d$ with $d \leq p-1$, $r \geq 2$ and $d - 2r \leq e$.

In this note we give the expected answer (yes) to a conjecture raised in [2, Conjecture 3.8]. The proof uses only the more elementary part of a theory introduced by D. Eisenbud and J. Harris in [4].

Let $X$ be a smooth, connected, complete curve and $L$ a line bundle on $X$. The Clifford index $\text{Cliff}(L)$ of $L$ is defined by $\text{Cliff}(L) = \text{deg } L - 2(h^0(X, L) - 1)$. The Clifford index $\text{Cliff}(X)$ of $X$ is defined by $\text{Cliff}(X) = \min\{ \text{Cliff}(L) : L \text{ is a line bundle on } X \text{ with } h^0(X, L) \geq 2 \text{ and } h^1(X, L) \geq 2 \}$. If $X$ has genus $p$ in the definition of $\text{Cliff}(X)$ we may use the condition “$\text{deg } L \leq p - 1$” instead of the condition “$h^1(X, L) \geq 2$”.

Fix an algebraically closed field $k$ with $\text{ch}(k) = 0$. Here we prove the following result:

THEOREM. A general $(e+2)$-gonal curve $X$ of genus $p \geq 2e+2$ has $\text{Cliff}(X) = e$. Furthermore $X$ has no $g^{r+2} _{e+2}$ with $r \geq 2$ and $e + 2r \leq p - 1$.

PROOF. We use induction on the genus. By Brill and Noether’s theory [3] the induction starts for example at the genus $2e + 2$, when a curve with general moduli has a $g^1_{d+2}$. Assume the result for a general $(e+2)$-gonal curve of genus $p - 1$. Fix integers $r, d$ with $r \geq 2$, $d \leq p - 1$ and $d - 2r \leq e$. Let $P$ be a point of $A$ at which a $g^{r+2} _{e+2}$ on $A$ ramifies. Let $E$ be an elliptic curve and $0 \in E$. Let $Y$ be the union of $A$ and $E$ with the points $P$ and $0$ identified. By the theory of admissible covers by J. Harris and D. Mumford [5, Proof of Theorem 5.6, Corollary 4, p. 71], $Y$ is in the closure in the moduli space of stable curves of genus $p$ of the set of smooth $(e+2)$-gonal curves. Let $B$ be a discrete valuation ring with closed point $o$ and generic point $t$. Let $f: Z \to B$ be flat and proper with $Z$ smooth, $f^{-1}(o) \cong Y$ and with $f^{-1}(t)$ an $(e+2)$-gonal smooth curve of genus $p$. By Lefschetz principle and the existence of $G^d _2$ on a suitable cover of the moduli space $M_p$, to obtain a contradiction we may assume that the geometric general fiber of $f$ (i.e. the extension of $f^{-1}(t)$ over the algebraic closure of $k(t)$) has a $g^* _{d}$. Certainly this $g^* _{d}$ is defined over a finite extension of $k(t)$. As in [4, §2] we find $j \geq 0$ and a $g^* _{d}$ limit on a semistable curve $Y'$ defined over $k$, $Y'$ union of $A$, a chain of rational curves $D_i$, $1 \leq i \leq j$, and $E$, with $A \cap D _i = \emptyset$ if $i > 1$, $A \cap D _1 = \{ P \}$, $D _i \cap D _s = \emptyset$ if and only if $|i - j| \leq 1$, $\text{card}(D _i \cap D _{i+1}) = 1$ for $1 \leq i \leq j - 1$, $E \cap D _s = \emptyset$ if $s < j$, $E \cap D _j = \{ 0 \}$, $E \cap A = \emptyset$ (unless $j = 0$). If $j = 0$, set $Y' := Y$. By definition of a $g^* _{d}$ limit,
its existence implies the existence of a \( g^d_d \) on \( A \). If \( d \leq p - 2 \), this contradicts the inductive assumption and the choice of \( A \). If \( d = p - 1 \), we may assume that the associated line bundle \( L \) on \( A \) has no base point. By Riemann and Roch we have \( h^1(A, L) = r - 1 \), and if \( r \geq 3 \), we obtain a contradiction by duality and the inductive assumption. Hence we may assume \( r = 2 \), \( d = p - 1 \) and \( L \) base-point free. We obtain \( p - 1 - 4 \leq e \) and by the choice of \( p > 2e + 2 \) we find \( e \leq 2 \). The only cases to check are the cases with \( e = 0 \) or \( e = 1 \), \( g = 6 \) or \( e = 2 \), and \( g = 7 \), which are well known. If \( e = 0 \), \( A \) is hyperelliptic. If \( g = 6 \), a general 3-gonal curve has no \( g^2_6 \) (necessarily not composed with a pencil) by a dimensional count. If \( g = 7 \), a general 4-gonal curve \( A \) cannot have a \( g^2_6 \) which maps \( A \) birationally onto \( C \subset \mathbb{P}^2 \), \( \deg C = 6 \), because \( C \) can have only double points (\( A \) has no \( g^1_3 \)), hence it has at least 3 \( g^1_4 \), contradicting a theorem of E. Arbarello and M. Cornalba [1, Theorem 2.6]. If \( A \) has a \( g^2_6 \) which maps \( A \) nonbirationally onto \( C \subset \mathbb{P}^2 \), \( C \) must be elliptic and \( A \) elliptic-hyperelliptic; again \( A \) cannot have infinite \( g^1_4 \). \( \square \)

REFERENCES


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