SUBHOMOGENEOUS AF C*-ALGEBRAS 
AND THEIR FUBINI PRODUCTS. II

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ABSTRACT. If C is a nuclear C*-subalgebra of a C*-algebra A, then we have 
C ⊗ D = (A ⊗ D) ∩ (C ⊗ B) for any C*-algebras B and D with D ⊂ B. Using 
this, we show that if A and B are AF algebras and A ⊗ F B = A ⊗ B, then either A 
or B must be subhomogeneous.

1. Introduction. Let A be an AF algebra. In the previous paper [6], we showed that 
if A ⊗ F B = A ⊗ B for any C*-algebra B, then A is subhomogeneous. More 
precisely, our arguments in [6] show that if A is not subhomogeneous, then 
A ⊗ F B(H) ⊃ A ⊗ B(H). The C*-algebra B(H) is, of course, not subhomoge-
neous. The purpose of this note is to show that if A and B are AF algebras neither 
of which are subhomogeneous, then A ⊗ F B ⊃ A ⊗ B. All notation and terminology 
follow those of the previous paper [6].

2. Intersection results for C*-tensor products. Let A (respectively B) be a 
C*-algebra with C*-subalgebras A₁ and A₂ (respectively B₁ and B₂). Then, it is 
clear that 

\[
(2.1) \quad (A_1 \cap A_2) \otimes (B_1 \cap B_2) \subseteq (A_1 \otimes B_1) \cap (A_2 \otimes B_2). 
\]

Wassermann [7] raised the question whether the equality in (2.1) holds or not, and 
Huruya [3] and the author [5] gave negative answers. The following theorem gives a 
sufficient condition for which the equality holds in (2.1) when A₁ ⊆ A₂ and 
B₁ ⊆ B₂.

THEOREM 2.1. Let A and C be C*-algebras with C ⊂ A. Assume that the pair 
(A, C) satisfies the following condition:

\[
(2.2) \quad \exists \{ \pi_λ : \lambda \in \Lambda \} \text{ of completely bounded linear maps from A into C such that } \sup_λ \|\pi_λ\|_{CB} < \infty \text{ and } \\
\lim_\lambda \|\pi_λ(x) - x\| = 0 \text{ for } x \in C.
\]

Then, we have C ⊗ D = (A ⊗ D) ∩ (C ⊗ B) for any pair (B, D) of C*-algebras 
with D ⊂ B.
Proof. Let $x \in C \otimes B$ and $\varepsilon > 0$ be given. Then, there exists an element $\sum_{i=1}^{n} a_i \otimes b_i$ in the algebraic tensor product $C \otimes B$ of $C$ and $B$ such that

$$\left\| x - \sum_{i=1}^{n} a_i \otimes b_i \right\| < \varepsilon / (M + 1),$$

where $M = \sup_{\lambda} \{ \|\pi_{\lambda}\|_{C \otimes B} \}$. Now, since $\pi_{\lambda} \otimes 1: A \otimes B \to C \otimes B$ is a bounded linear map with $\|\pi_{\lambda} \otimes 1\| \leq M$, we have

$$\| (\pi_{\lambda} \otimes 1)(x) - x \| \lesssim \|\pi_{\lambda} \otimes 1\!* \sum_{i=1}^{n} \|\pi_{\lambda}(a_i) - a_i\| \|b_i\| \| \sum_{i=1}^{n} a_i \otimes b_i - x \|. \tag{2.3}$$

If we choose $\lambda_0$ such that $\lambda \geq \lambda_0$ implies

$$\|\pi_{\lambda}(a_i) - a_i\| < \varepsilon / 2n \|b_i\| \quad \text{for } i = 1, 2, \ldots, n,$$

then we have

$$\| (\pi_{\lambda} \otimes 1)(x) - x \| < M \frac{\varepsilon}{2(M + 1)} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2(M + 1)} = \varepsilon$$

for $\lambda \geq \lambda_0$ from (2.3).

Now, if $x \in (C \otimes B) \cap (A \otimes D)$, then $x \in C \otimes B$ implies $x = \lim_{\lambda} (\pi_{\lambda} \otimes 1)(x)$ from the preceding paragraph, and $x \in A \otimes D$ implies that $(\pi_{\lambda} \otimes 1)(x) \in C \otimes D$ because $\pi_{\lambda}$ maps from $A$ into $C$. Hence, we have

$$x = \lim_{\lambda} (\pi_{\lambda} \otimes 1)(x) \in C \otimes D.$$

**Corollary 2.2.** If $C$ is a nuclear $C^*$-subalgebra of a $C^*$-algebra $A$, then we have $C \otimes D = (A \otimes D) \cap (C \otimes B)$ for any pair $(B, D)$ of $C^*$-algebras with $D \subset B$.

**Proof.** Note that the pair $(A, C)$ satisfies condition (2.2) as in the proof of [1, Corollary 1].

3. **Fubini products of $AF$ $C^*$-algebras.** For the sake of convenience, put $M_0 = M \cap \mathcal{K}(H) = \{(x_n); \ x_n \in M_n \text{ and } \lim_n \|x_n\| = 0\}$ (see [6, Example 2.1]). First, we show that $M_0 \otimes F M_0 \supsetneq M_0 \otimes M_0$.

**Example 3.1.** We have $M_0 \otimes F M_0 \supsetneq M_0 \otimes M_0$. Indeed,

$$M_0 \otimes M_0 = (B(H) \otimes M_0) \cap (M_0 \otimes B(H))$$

$$\supsetneq F(\mathcal{B}(H), M_0, B(H) \otimes B(H)) \cap F(M_0, B(H), B(H) \otimes B(H))$$

$$= F(M_0, M_0, B(H) \otimes B(H)) = M_0 \otimes F M_0,$$

where the first equality follows from Corollary 2.2, and the proper inclusion follows from [4, Example 11].

**Proposition 3.2.** Let $A$, $B$, $C$ and $D$ be nuclear $C^*$-algebras with $C \subset A$ and $D \subset B$. If $A \otimes_F B = A \otimes B$, then we have $C \otimes_F D = C \otimes D$. 

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Proof. Let $A_0$ (respectively $B_0$) be an injective $C^*$-algebra containing $A$ (respectively $B$). Then we have $C \otimes_F D = F(C, D, A_0 \otimes B_0) \subset F(A, B, A_0 \otimes B_0) = A \otimes B$. Hence, it follows that

\[
C \otimes_F D \subset F(C, D, A \otimes B) = F(A, D, A \otimes B) \cap (C, B, A \otimes B)
\]

(3.1)

\[
= (A \otimes D) \cap (C \otimes B)
\]

(3.2)

\[
= C \otimes D,
\]

where the equality (3.1) follows from [2, Theorem 3.4] and the equality (3.2) follows from Corollary 2.2.

Theorem 3.3. Let $A$ and $B$ be AF $C^*$-algebras. Then the following are equivalent:

(i) $A \otimes_F B = A \otimes B$.

(ii) Either $A$ or $B$ is subhomogeneous.

Proof. It suffices to show the implication (i) $\Rightarrow$ (ii). Assume that neither $A$ nor $B$ are subhomogeneous. Then, $A$ (respectively $B$) contains a $C^*$-subalgebra $C$ (respectively $D$) which is isomorphic to the $C^*$-algebra $M_0$ by [6, Theorem 2.3]. Now, we have a contradiction by Example 3.1 and Proposition 3.2.

References


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