PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 2, June 1986

POWER SERIES AND NONNORMAL FUNCTIONS

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ABSTRACT. We construct a sequence $\{a_n\}$ of positive numbers such that $a_n \to 0$, $\sum |a_n - a_{n-1}| < \infty$, and the function $f(z) = \sum a_n z^n$ is not a normal function. This answers a question raised by the second author (Proc. Amer. Math. Soc. **85** (1982), 335–341).

1. Introduction. Let $D = \{z: |z| < 1\}$, $C = \{z: |z| = 1\}$, and $f(z) = \sum a_n z^n$ be a function analytic in D. The function f is said to be a normal function if the family of functions $\{f(z; a, \theta) = f(e^{i\theta}(z + a)/(1 + \bar{a}z)): a \in D,\ \theta \in [0, 2\pi)\}$ is a normal family in the sense of Montel, that is, each sequence of functions in the family contains a subsequence which converges uniformly on each compact subset of D either to an analytic function or to ∞.

In [**1**], the second author proved the following criterion for normal functions.

THEOREM L. *Let $\{a_n\}$ be a sequence of complex numbers such that both $\sum_{n=1}^{\infty} |a_n - a_{n-1}| < \infty$ and $a_n \nrightarrow 0$. Then $\sum_{n=0}^{\infty} a_n z^n$ is a normal function.*

In the same paper [**1**, Remark 2, p. 338] the question was raised as to whether Theorem L is valid if the condition "$a_n \nrightarrow 0$" is replaced by ""$a_n \to 0$". In a later paper [**2**, Theorem 2, p. 310] this question was answered in the negative by an example in which all the terms a_n were real numbers, but many of the terms a_n were negative. The purpose of this note is to present a basically different example to show that Theorem L is not valid even when the condition "$a_n \nrightarrow 0$" is replaced by the condition "$a_n \to 0$ and $a_n > 0$ for each n".

THEOREM. *There exists a sequence $\{a_n\}$ of positive numbers such that $a_n \to 0$, $a_n > 0$ for each n, $\sum |a_n - a_{n-1}| < \infty$, and the function $f(z) = \sum a_n z^n$ is not a normal function.*

PROOF OF THE THEOREM. Let $N_0 = -1$, let $d = 2^{12} e^{6\pi}$, and let $\{N_j: j = 1, 2, 3, \ldots\}$ be a sequence of positive integers, each of which is divisible by 4, satisfying

$$N_{j+1} > (1 + N_j)^2 \quad \text{for } j > 0 \text{ and } N_1 > d^4. \tag{1}$$

Let $\theta_j = 2\pi/N_j$ and $r_j = 1 - \theta_j$ for $j > 0$. For each $j > 0$ and $k \in \{2, 3, 4\}$, let

$$C_{j,k} = \sum_{n=((K-1)N_j/4)+1}^{kN_j/4} r_j^n \cos(n\theta_j) \quad \text{and} \quad S_{j,k} = \sum_{n=((k-1)N_j/4)+1}^{kN_j/4} r_j^n \sin(n\theta_j).$$

Received by the editors October 18, 1984.
1980 *Mathematics Subject Classification*. Primary 30D45.
Key words and phrases. Normal function.

Further, let

$$C_1 = \sum_{n=0}^{N_1/4} N_1^{-1/2} r_1^n \cos(n\theta_1) \quad \text{and} \quad S_1 = \sum_{n=0}^{N_1/4} N_1^{-1/2} r_1^n \sin(n\theta_1),$$

let $b_1 = -N_1^{1/2} C_1/C_{1,2}$ and for $0 \le n \le N_1/4$ let $a_n = N_1^{-1/2}$. We claim that we can determine sequences $\{a_n\}, \{b_j\}, \{x_j\}$, and $\{y_j\}$ which satisfy all of the following conditions (2), (3), (4), and (5) simultaneously:

$$C_j = \sum_{n=0}^{N_j/4} a_n r_j^n \cos(n\theta_j) \quad \text{and} \quad S_j = \sum_{n=0}^{N_j/4} a_n r_j^n \sin(n\theta_j), \tag{2}$$

$$N_j^{1/2} C_j + b_j C_{j,2} = 0, \tag{3}$$

$$\begin{cases} x_j C_{j,3} + y_j C_{j,4} = 0, \\ x_j S_{j,3} + y_j S_{j,4} = -(N_j^{1/2} S_j + b_j S_{j,2}), \end{cases} \tag{4}$$

$$a_n = \begin{cases} N_j^{-1/2}, & \text{for } N_{j-1} + 1 \le n \le N_j/4, \\ b_j N_j^{-1/2}, & \text{for } N_j/4 < n \le N_j/2, \\ x_j N_j^{-1/2}, & \text{for } N_j/2 < n \le 3N_j/4, \\ y_j N_j^{-1/2}, & \text{for } 3N_j/4 < n \le N_j. \end{cases} \tag{5}$$

We will show that we can define the terms in these sequences inductively. We have already defined b_1, a_n for $0 \le n \le N_1/4$, C_1, S_1, and (2), (3), and (5) are satisfied for these terms with $j = 1$. Now for each j, (4) is a system of equations with the unknowns x_j and y_j for which the determinant of the coefficients, $\Delta_j = C_{j,3} S_{j,4} - C_{j,4} S_{j,3}$, is positive, since each of these coefficients is negative except for $C_{j,4}$, which is postive. Hence, if we assume that all terms in (4) are defined except for x_j and y_j, we see that (4) determines a unique solution for x_j and y_j. For $j = 1$, such a solution pair x_i and y_1 exists. Now (5) defines a_n for $0 \le n \le N_1$.

Now suppose that b_k, x_k, and y_k are all known for all $k \le j$, and suppose that a_n is known for all $n \le N_j$. Then we can define $a_n = N_{j+1}^{-1/2}$ for $N_j + 1 \le n \le N_{j+1}/4$, which means that (2) now defines both C_{j+1} and S_{j+1}, (3) determines b_{j+1}, (4) determines both x_{j+1} and y_{j+1}, and (5) defines a_n for $0 \le n \le N_{j+1}$. Thus, we can proceed inductively to define the full sequences $\{a_n\}, \{b_j\}, \{x_j\}$, and $\{y_j\}$.

Next, we claim both

$$e^{-2\pi}/80 < b_j < 4\pi e^{2\pi} \quad \text{for each } j, \tag{6}$$

and

$$1/d < x_j < y_j < d \qquad \text{for each } j. \tag{7}$$

To prove these, we again proceed inductively. First, we remark that

$$e^{-2\pi}/2 < (1 - (2\pi/N_j))^{N_j} = (1 - \theta_j)^{N_j} \le (1 - \theta_j)^n = r_j^n < 1$$

for $1 \le n \le N_j$. Also, we note the standard formulas

$$\sum_{n=p}^{q} \cos n\theta = [\sin((2q+1)\theta/2) - \sin((2p-1)\theta/2)]/(2\sin\theta/2),$$

and

$$\sum_{n=p}^{q} \sin n\theta = [\cos((2p-1)\theta/2) - \cos((2q+1)\theta/2)]/(2\sin\theta/2),$$

and also that $1/\theta < 1/(2\sin\theta/2) < 2/\theta$ for $0 < \theta < \pi/2$. Thus, letting $\theta_j = \theta$ and letting p and q be consecutive multiples of $N_j/4$, we obtain

$$N_j e^{-2\pi}/(4\pi) < |C_{j,k}|, |S_{j,k}| < 2N_j/\pi \quad \text{for all } j \text{ and } k \in \{2,3,4\}, \tag{8}$$

and also $N_1 e^{-2\pi}/(4\pi) < N_1^{1/2} S_1, N_1^{1/2} C_1 < 2N_1/\pi$. Thus, we have

$$b_1 = -N^{1/2} C_1/C_{1,2} < 8e^{2\pi} < 4\pi e^{2\pi}.$$

Actually, it is easy to see that $N_1^{1/2} C_1 > |C_{1,2}|$ by pairing cosine terms of equal absolute value, so $b_1 > 1$. Thus, we have $1 < b_1 < 4\pi e^{2\pi}$. From (4) and Cramer's Rule, we have

$$x_j = C_{j,4}(N_j^{1/2} S_j + b_j S_{j,2})/\Delta_j \quad \text{and} \quad y_j = -C_{j,3}(N_j^{1/2} S_j + b_j S_{j,2})/\Delta_j.$$

Note also that $|C_{j,3}| > C_{j,4}$, so $0 < x_j < y_j$ for each j. From (8), we have

$$N_j^2 e^{-4\pi}/(8\pi^2) < \Delta_j < 8N_j^2/\pi^2 \quad \text{for each } j. \tag{9}$$

It follows from (8) and (9) and the estimates on b_1 that

$$1/d < e^{-4\pi}/64 < x_1 < y_1 < 32e^{4\pi}(1 + 4\pi e^{2\pi}) < d.$$

To get estimates on b_j, x_j, and y_j for $j > 1$, we proceed inductively. Suppose that (6) and (7) hold for all $j < p$, where $p > 1$. Then

$$C_p = \sum_{n=0}^{N_{p-1}/4} a_n r_p^n \cos n\theta_p + \sum_{n=N_{p-1}+1}^{N_p/4} N_p^{-1/2} r_p^n \cos n\theta_p,$$

and

$$S_p = \sum_{n=0}^{N_{p-1}/4} a_n r_p^n \sin n\theta_p + \sum_{n=N_{p-1}+1}^{N_p/4} N_p^{-1/2} r_p^n \sin n\theta_p.$$

By the inductive hypothesis, we have that $a_n < d/N_1^{1/2}$ for all $n \le N_{p-1}$ and hence, using (1), we have

$$\sum_{n=0}^{N_{p-1}} a_n < (N_{p-1}+1)d/N_1^{1/2} < N_{p-1}/d.$$

Further, using the same estimates that yielded (8) and noting that $N_{p-1} < N_p/8$, we have

$$\begin{aligned}(7/8)N_p^{1/2} e^{-2\pi}/4\pi &< N_p^{-1/2} \sum_{n=N_{p-1}+1}^{N_p/4} r_p^n \cos n\theta_p \\ &< N_p^{-1/2}(2N_p/\pi) = 2N_p^{1/2}/\pi,\end{aligned}$$

and the same inequality is valid if $\cos n\theta_p$ is replaced by $\sin n\theta_p$. Thus, we have

$$7N_p^{1/2} e^{-2\pi}/32\pi < C_p, S_p < (N_{p-1}/d) + (2N_p^{1/2}/\pi) < N_p^{1/2}. \tag{10}$$

Now, from (3), (8), and (10), we have

$$e^{-2\pi}/80 < 7e^{-2\pi}/64 < b_p = -N_p^{1/2}C_p/C_{p,2} < 4\pi e^{2\pi}$$

and also, from (4), (8), (9), and (10) we have

$$1/d < e^{-4\pi}/1280 < x_p < y_p < 16\pi(1+8e^{2\pi})e^{4\pi} < d.$$

Thus (6) and (7) are established.

To complete the proof, let $f(z) = \sum a_n z^n$. We note that by (5), (6), and (7), we have $a_n \geq 1/(dN_j^{1/2})$ for $N_{j-1} < n \leq N_j$, so

$$\begin{aligned}\liminf_{x\to 1-} f(x) &\geq \sum_{j=1}^{\infty} \sum_{n=N_j+1}^{N_{j+1}} 1/(dN_{j+1}^{1/2}) = (1/d)\sum_{j=1}^{\infty}(N_{j+1}-N_j)N_{j+1}^{-1/2} \\ &> (1/2d)\sum_{j=1}^{\infty} N_{j+1}^{1/2} = \infty.\end{aligned}$$

Thus, $f(z)$ has the radial limit ∞ at $z = 1$. If f were a normal function, then f would have the angular limit ∞ at $z = 1$ (see [**3**]). But let $z_j = r_j e^{i\theta_j}$, where r_j and θ_j are defined as before for each j. If we consider the triangle with vertices at $1, z_j$, and $e^{i\theta_j}$, we see that $|e^{i\theta_j} - z_j| = \theta_j$ and $|e^{i\theta_j} - 1| = 2\sin(\theta_j/2)$ and the angle at the vertex $e^{i\theta_j}$ is $(\pi - \theta_j)/2$. Thus, we have that the sequence $\{z_j\}$ approaches the point 1 at an angle close to $\pi/4$ from the radius, and so $\{z_j\}$ approaches $z = 1$ nontangentially.

On the other hand, if we set $f_p(z) = \sum_{n=0}^{N_p} a_n z^n$ and $g_p(z) = f(z) - f_p(z)$, we have that both the real part of $f_p(z_p)$ and the imaginary part of $f_p(z_p)$ are zero as a result of (2), (3), (4), and (5). We claim that $g_p(z_p)$ is uniformly bounded. For we have

$$\begin{aligned}|g_p(z_p)| &= \left|\sum_{n=N_p+1}^{\infty} a_n z_p^n\right| \leq (d/N_{p+1}^{1/2})(1/(1-r_p)) \\ &= (d/N_{p+1}^{1/2})(N_p/2\pi) < d/2\pi.\end{aligned}$$

It follows that $|f(z_j)| \leq d/2\pi$ for each j. Thus f is not a normal function.

Finally, we note that both $a_n \to 0$ (since $0 < a_n < d/N_j^{-1/2}$ for $N_{j-1} < n \leq N_j$) and

$$\sum_{n=1}^{\infty} |a_n - a_{n-1}| < \sum_{j=1}^{\infty} 4dN_j^{-1/2} < (4d)\sum_{j=1}^{\infty}(N_1^{-1/2})^j < \infty,$$

since (1) implies that $N_j > N_1^j$. This completes the proof of the Theorem.

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