ABSTRACT. We construct a sequence \( \{a_n\} \) of positive numbers such that
\( a_n \to 0, \sum |a_n - a_{n-1}| < \infty \), and the function \( f(z) = \sum a_n z^n \) is not a normal function. This answers a question raised by the second author (Proc. Amer. Math. Soc. 85 (1982), 335–341).

1. Introduction. Let \( D = \{z: |z| < 1\} \), \( C = \{z: |z| = 1\} \), and \( f(z) = \sum a_n z^n \) be a function analytic in \( D \). The function \( f \) is said to be a normal function if the family of functions \( \{f(z; a, \theta) = f(e^{i\theta}(z + a)/(1 + az)): a \in D, \theta \in [0, 2\pi)\} \) is a normal family in the sense of Montel, that is, each sequence of functions in the family contains a subsequence which converges uniformly on each compact subset of \( D \) either to an analytic function or to \( \infty \).

In [1], the second author proved the following criterion for normal functions.

**Theorem L.** Let \( \{a_n\} \) be a sequence of complex numbers such that both \( \sum_{n=1}^{\infty} |a_n - a_{n-1}| < \infty \) and \( a_n \to 0 \). Then \( \sum_{n=0}^{\infty} a_n z^n \) is a normal function.

In the same paper [1, Remark 2, p. 338] the question was raised as to whether Theorem L is valid if the condition “\( a_n \to 0 \)” is replaced by “\( a_n \to 0 \)” in a later paper [2, Theorem 2, p. 310] this question was answered in the negative by an example in which all the terms \( a_n \) were real numbers, but many of the terms \( a_n \) were negative. The purpose of this note is to present a basically different example to show that Theorem L is not valid even when the condition “\( a_n \to 0 \)” is replaced by the condition “\( a_n \to 0 \)” and \( a_n > 0 \) for each \( n \).”

**Theorem.** There exists a sequence \( \{a_n\} \) of positive numbers such that \( a_n \to 0, a_n > 0 \) for each \( n \), \( \sum |a_n - a_{n-1}| < \infty \), and the function \( f(z) = \sum a_n z^n \) is not a normal function.

**Proof of the Theorem.** Let \( N_0 = -1 \), let \( d = 2^{12} e^{6\pi} \), and let \( \{N_j: j = 1, 2, 3, \ldots\} \) be a sequence of positive integers, each of which is divisible by \( 4 \), satisfying

\[
N_{j+1} > (1 + N_j)^2 \quad \text{for } j > 0 \text{ and } N_1 > d^4.
\]

Let \( \theta_j = 2\pi/N_j \) and \( r_j = 1 - \theta_j \) for \( j > 0 \). For each \( j > 0 \) and \( k \in \{2, 3, 4\} \), let

\[
C_{j,k} = \sum_{n=((K-1)N_j/4)+1}^{kN_j/4} r_j^n \cos(n\theta_j) \quad \text{and} \quad S_{j,k} = \sum_{n=((k-1)N_j/4)+1}^{kN_j/4} r_j^n \sin(n\theta_j).
\]
Further, let
\[ C_1 = \sum_{n=0}^{N_1/4} N_1^{-1/2} r_1^n \cos(n\theta_1) \quad \text{and} \quad S_1 = \sum_{n=0}^{N_1/4} N_1^{-1/2} r_1^n \sin(n\theta_1), \]

let \( b_1 = -N_1^{1/2} C_1/C_{1,2} \) and for \( 0 \leq n \leq N_1/4 \) let \( a_n = N_1^{-1/2} \). We claim that we can determine sequences \( \{a_n\}, \{b_j\}, \{x_j\}, \) and \( \{y_j\} \) which satisfy all of the following conditions (2), (3), (4), and (5) simultaneously:

\[
(2) \quad C_j = \sum_{n=0}^{N_j/4} a_n r_j^n \cos(n\theta_j) \quad \text{and} \quad S_j = \sum_{n=0}^{N_j/4} a_n r_j^n \sin(n\theta_j),
\]

(3) \[ N_j^{1/2} C_j + b_j C_{j,2} = 0, \]

(4) \[
\begin{cases}
  x_j C_{j,3} + y_j C_{j,4} = 0, \\
  x_j S_{j,3} + y_j S_{j,4} = -(N_j^{1/2} S_j + b_j S_{j,2}),
\end{cases}
\]

(5) \[ a_n = \begin{cases}
  N_j^{-1/2}, & \text{for } N_j - 1 + 1 \leq n \leq N_j/4, \\
  b_j N_j^{-1/2}, & \text{for } N_j/4 < n \leq N_j/2, \\
  x_j N_j^{-1/2}, & \text{for } N_j/2 < n \leq 3N_j/4, \\
  y_j N_j^{-1/2}, & \text{for } 3N_j/4 < n \leq N_j.
\end{cases} \]

We will show that we can define the terms in these sequences inductively. We have already defined \( b_1, a_n \) for \( 0 \leq n \leq N_1/4 \), \( C_1, S_1 \), and (2), (3), and (5) are satisfied for these terms with \( j = 1 \). Now for each \( j \), (4) is a system of equations with the unknowns \( x_j \) and \( y_j \) for which the determinant of the coefficients, \( \Delta_j = C_{j,3} S_{j,4} - C_{j,4} S_{j,3} \), is positive, since each of these coefficients is negative except for \( C_{j,4} \), which is positive. Hence, if we assume that all terms in (4) are defined except for \( x_j \) and \( y_j \), we see that (4) determines a unique solution for \( x_j \) and \( y_j \). For \( j = 1 \), such a solution pair \( x_1 \) and \( y_1 \) exists. Now (5) defines \( a_n \) for \( 0 \leq n \leq N_1 \).

Now suppose that \( b_k, x_k, \) and \( y_k \) are all known for all \( k < j \), and suppose that \( a_n \) is known for all \( n \leq N_j \). Then we can define \( a_n = N_j^{-1/2} \) for \( N_j + 1 \leq n \leq N_j + 1/4 \), which means that (2) now defines both \( C_{j+1} \) and \( S_{j+1} \), (3) determines \( b_{j+1} \), (4) determines both \( x_{j+1} \) and \( y_{j+1} \), and (5) defines \( a_n \) for \( 0 \leq n \leq N_j + 1 \). Thus, we can proceed inductively to define the full sequences \( \{a_n\}, \{b_j\}, \{x_j\}, \) and \( \{y_j\} \).

Next, we claim both
\[
(6) \quad e^{-2\pi/80} < b_j < 4\pi e^{2\pi} \quad \text{for each } j,
\]

and
\[
(7) \quad 1/d < x_j < y_j < d \quad \text{for each } j.
\]

To prove these, we again proceed inductively. First, we remark that
\[
e^{-2\pi/2} < (1 - (2\pi/N_j))^N_j = (1 - \theta_j)^N_j \leq (1 - \theta_j)^n = r_j^n < 1 \]

for \( 1 \leq n \leq N_j \). Also, we note the standard formulas
\[
\sum_{n=p}^{q} \cos n\theta = \frac{\sin((2q + 1)\theta/2) - \sin((2p - 1)\theta/2))}{2\sin \theta/2},
\]

\[
\sum_{n=p}^{q} \sin n\theta = \frac{\sin((2q + 1)\theta/2) - \sin((2p - 1)\theta/2))}{2\sin \theta/2}.
\]
and
\[
\sum_{n=p}^{q} \sin n\theta = \frac{\cos((2p-1)\theta/2) - \cos((2q+1)\theta/2)}{(2\sin \theta/2)},
\]
and also that \(1/\theta < 1/(2\sin \theta/2) < 2/\theta\) for \(0 < \theta < \pi/2\). Thus, letting \(\theta_j = \theta\) and letting \(p\) and \(q\) be consecutive multiples of \(N_j/4\), we obtain
\[
N_j e^{-2\pi/(4\pi)} < |C_{j,k}|, |S_{j,k}| < 2N_j/\pi \quad \text{for all } j \text{ and } k \in \{2, 3, 4\},
\]
and also \(N_1 e^{-2\pi/(4\pi)} < N_1^{1/2} S_1, N_1^{1/2} C_1 < 2N_1/\pi\). Thus, we have
\[
b_1 = -N_1^{1/2} C_1/C_{1,2} < 8e^{2\pi} < 4\pi^{2\pi}.
\]
Actually, it is easy to see that \(N_1^{1/2} C_1 > |C_{1,2}|\) by pairing cosine terms of equal absolute value, so \(b_1 > 1\). Thus, we have \(1 < b_1 < 4\pi^{2\pi}\). From (4) and Cramer's Rule, we have
\[
x_j = C_{j,4}(N_j^{1/2} S_j + b_j S_{j,2})/\Delta_j \quad \text{and} \quad y_j = -C_{j,3}(N_j^{1/2} S_j + b_j S_{j,2})/\Delta_j.
\]
Note also that \(|C_{j,3}| > C_{j,4}\), so \(0 < x_j < y_j\) for each \(j\). From (8), we have
\[
N_j^2 e^{-4\pi/(8\pi^2)} < \Delta_j < 8N_j^2/\pi^2 \quad \text{for each } j.
\]
It follows from (8) and (9) and the estimates on \(b_1\) that
\[
1/d < e^{-4\pi/64} < x_1 < y_1 < 32e^{4\pi(1 + 4\pi^{2\pi})} < d.
\]
To get estimates on \(b_j, x_j,\) and \(y_j\) for \(j > 1\), we proceed inductively. Suppose that (6) and (7) hold for all \(j < p\), where \(p > 1\). Then
\[
C_p = \sum_{n=0}^{N_p-1/4} a_n r_p^n \cos n\theta_p + \sum_{n=N_p/4}^{N_p/4} N_p^{-1/2} r_p^n \cos n\theta_p,
\]
and
\[
S_p = \sum_{n=0}^{N_p-1/4} a_n r_p^n \sin n\theta_p + \sum_{n=N_p/4}^{N_p/4} N_p^{-1/2} r_p^n \sin n\theta_p.
\]
By the inductive hypothesis, we have that \(a_n < d/N_1^{1/2}\) for all \(n \leq N_{p-1}\) and hence, using (1), we have
\[
\sum_{n=0}^{N_{p-1}} a_n < (N_{p-1} + 1)d/N_1^{1/2} < N_{p-1}/d.
\]
Further, using the same estimates that yielded (8) and noting that \(N_{p-1} < N_p/8\), we have
\[
(7/8) N_p^{1/2} e^{-2\pi/4\pi} < N_p^{-1/2} \sum_{n=N_p/4}^{N_p/4} r_p^n \cos n\theta_p < N_p^{-1/2} (2N_p/\pi) = 2N_p^{1/2}/\pi,
\]
and the same inequality is valid if \(\cos n\theta_p\) is replaced by \(\sin n\theta_p\). Thus, we have
\[
7N_p^{1/2} e^{-2\pi/32\pi} < C_p, S_p < (N_{p-1}/d) + (2N_p^{1/2}/\pi) < N_p^{1/2}.
\]
Now, from (3), (8), and (10), we have
\[ e^{-2\pi/80} < 7e^{-2\pi/64} < b_p = -N_p^{1/2}C_p/C_{p,2} < 4\pi e^{2\pi} \]
and also, from (4), (8), (9), and (10) we have
\[ 1/d < e^{-4\pi/1280} < x_p < y_p < 16\pi(1 + 8e^{2\pi})e^{4\pi} < d. \]
Thus (6) and (7) are established.

To complete the proof, let \( f(z) = \sum a_nz^n \). We note that by (5), (6), and (7), we have \( a_n \geq 1/(dN_j^{1/2}) \) for \( N_{j-1} < n \leq N_j \), so
\[
\liminf_{x \to 1-} f(x) \geq \sum_{j=1}^{\infty} \sum_{n=N_j+1}^{N_{j+1}} 1/(dN_{j+1}^{1/2}) = (1/d) \sum_{j=1}^{\infty} (N_{j+1} - N_j)N_j^{-1/2}
\]
\[ > (1/2d) \sum_{j=1}^{\infty} N_j^{1/2} = \infty. \]
Thus, \( f(z) \) has the radial limit \( \infty \) at \( z = 1 \). If \( f \) were a normal function, then \( f \) would have the angular limit \( \infty \) at \( z = 1 \) (see [3]). But let \( z_j = r_je^{i\theta_j} \), where \( r_j \) and \( \theta_j \) are defined as before for each \( j \). If we consider the triangle with vertices at \( 1, z_j, \) and \( e^{i\theta_j} \), we see that \( |e^{i\theta_j} - z_j| = \theta_j \) and \( |e^{i\theta_j} - 1| = 2\sin(\theta_j/2) \) and the angle at the vertex \( e^{i\theta_j} \) is \((\pi - \theta_j)/2 \). Thus, we have that the sequence \( \{z_j\} \) approaches the point 1 at an angle close to \( \pi/4 \) from the radius, and so \( \{z_j\} \) approaches \( z = 1 \) nontangentially.

On the other hand, if we set \( f_p(z) = \sum_{n=0}^{N_p} a_nz^n \) and \( g_p(z) = f(z) - f_p(z) \), we have that both the real part of \( f_p(z) \) and the imaginary part of \( f_p(z) \) are zero as a result of (2), (3), (4), and (5). We claim that \( g_p(z_p) \) is uniformly bounded. For we have
\[
|g_p(z_p)| = \left| \sum_{n=N_p+1}^{\infty} a_nz_p^n \right| \leq (d/N_{p+1}^{1/2})(1/(1 - r_p))
\]
\[ = (d/N_{p+1}^{1/2})(N_p/2\pi) < d/2\pi. \]
It follows that \( |f(z_j)| \leq d/2\pi \) for each \( j \). Thus \( f \) is not a normal function.

Finally, we note that both \( a_n \to 0 \) (since \( 0 < a_n < d/N_j^{-1/2} \) for \( N_{j-1} < n \leq N_j \)) and
\[ \sum_{n=1}^{\infty} |a_n - a_{n-1}| < \sum_{j=1}^{\infty} 4dN_j^{-1/2} < (4d) \sum_{j=1}^{\infty} (N_1^{-1/2})j < \infty, \]
since (1) implies that \( N_j > N_1^j \). This completes the proof of the Theorem.

REFERENCES


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