ON THE OSCILLATION OF ALMOST-PERIODIC STURM-LIOUVILLE OPERATORS WITH AN ARBITRARY COUPLING CONSTANT1
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ABSTRACT. In this paper we characterize those (Bohr) almost periodic functions \( V \) on \( \mathbb{R} \) for which the Sturm-Liouville equations

\[-y'' + \lambda V(x)y = 0, \quad x \in \mathbb{R},\]

are oscillatory at \( \pm \infty \) for every real \( \lambda \neq 0 \), or, equivalently, for which there exists a real \( \lambda \neq 0 \) such that the equation has a positive solution on \( \mathbb{R} \).

1. Introduction. In the study of the disconjugacy domain \( D \) (see [4]) of a linear second order differential equation with two parameters \( \alpha, \beta \),

\[-y'' + (\alpha - \beta V(x))y = 0, \quad x \in \mathbb{R},\]

one is inevitably drawn into the important case when \( D \subseteq \{(\alpha, \beta): \alpha > 0\} \cup \{(0,0)\} \) which occurs, for example, when \( V \) is periodic and has mean value equal to zero, as in Hill's equation. We recall that the equation

\[-y'' + V(x)y = 0, \quad x \in \mathbb{R},\]

is said to be disconjugate on \( \mathbb{R} \) provided every one of its (nonidentically zero) solutions has at most one zero in \( (-\infty, \infty) \). This is equivalent to the fact that there exists a solution \( y(x) > 0 \) for \( x \in \mathbb{R} \). \( D \) is then the collection of \( (\alpha, \beta) \in \mathbb{R}^2 \) for which (1.1) is disconjugate on \( \mathbb{R} \). If \( D \) is contained in the right-half plane of the parameter space \( \mathbb{R}^2 \), as above, it follows that

\[-y'' + \lambda V(x)y = 0\]

is oscillatory (at both ends \( \pm \infty \)) for every real \( \lambda \neq 0 \). General conditions on \( V \) for which this behavior is realized may be found in our recent monograph [3]. The case when \( V \) is a (Bohr) almost periodic function was considered by Markus and Moore [4]. In this case we show that it is possible to characterize those almost-periodic \( V \) for which (1.2) is oscillatory at \( \pm \infty \) for every real \( \lambda \neq 0 \). The results used to obtain this characterization are drawn from oscillation theory, in particular, results of Moore [5] and Wintner [7] are central to our investigations. A by-product of our techniques is that various classes of generalized almost-periodic functions such as those considered by Weyl and Besicovitch (see [1] for more details) can also be treated.

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2. Basic results and terminology. In the sequel, \( M\{V(x)\} \) will denote the mean-value of an almost periodic function,

\[
M\{V(x)\} = \lim_{T \to \infty} \frac{1}{T} \int_0^T V(s) \, ds.
\]

We recall that a.p. functions have a mean value that is uniform, in the sense that

\[
M\{V(x)\} = \lim_{T \to \infty} \frac{1}{T} \int_a^{a+T} V(s) \, ds
\]

uniformly for \( a \in \mathbb{R} \) (see [1, 2]). For a given \( V, v \) will denote some, generally unspecified, indefinite integral of \( V \).

3. The main theorem.

**Theorem 3.1.** Let \( V \neq 0 \) be an almost periodic function. Then a necessary and sufficient condition for (1.2) to be oscillatory at \( \pm \infty \) for every real \( \lambda \neq 0 \) is that \( M\{V(x)\} = 0 \).

**Corollary 3.2.** The following statements are equivalent:

(i) The equation (1.2) is oscillatory at \( \pm \infty \) for every real \( \lambda \neq 0 \).

(ii) These exist finite numbers \( \lambda^+ > 0 \) and \( \lambda^- < 0 \), such that (1.2) is oscillatory at \( \pm \infty \) for every \( \lambda \in (\lambda^-, \lambda^+) \), \( \lambda \neq 0 \).

(iii) \( M\{V(x)\} = 0 \).

**Proof.** That (i)\(\iff\)(iii) and (i)\(\implies\)(ii) is clear. That (ii)\(\implies\)(i) follows from the convexity of the disconjugacy domain [4].

The case \( V \) purely periodic with \( M\{V(x)\} = 0 \) of Theorem 3.1 can be found in Staněk [6]. However, this case actually follows directly from [4, Theorems 2 and 6].

**Proof of Theorem 3.1 (Sufficiency).** This was essentially shown in [4, Theorem 2]. Another proof may also be found in W. Coppel’s monograph *Disconjugacy* [Theorem 14].

**(Necessity).** We will show that whenever \( M\{V(x)\} \neq 0 \), there exists a value of \( \lambda \in \mathbb{R}, \lambda \neq 0 \), for which (1.2) is disconjugate on \( \mathbb{R} \). It will follow from this that (1.2) will be oscillatory for every real \( \lambda \neq 0 \), only if \( M\{V(x)\} = 0 \).

To this end let \( M\{V(x)\} = m \neq 0 \) and consider the single differential equation in the two real parameters \( \mu, \nu \):

\[
y'' + (-\nu + \mu V(x))y = 0
\]

on \([0, \infty)\). (Note that nonoscillation on \([0, \infty)\) implies disconjugacy on \([0, \infty)\) and so on \((-\infty, \infty)\) by results in [4].) Then (3.1) may be rewritten as

\[
y'' + (-\alpha + \beta V^*(x))y = 0,
\]

where \( \beta = \mu, \alpha = \nu - m\mu \) and \( M\{V^*(x)\} = 0 \). (Let \( \alpha > 0, \beta \neq 0 \).) We now make the transformation \( y = z \exp(-x\sqrt{\alpha}) \) and \( t = (1/2\sqrt{\alpha}) \exp(2\sqrt{\alpha}x) \). This leads us to the equation

\[
z'' + \beta e^{-4\sqrt{\alpha}x}V(x)z = 0
\]
and \( f(t) = \beta V(x) \exp(-4\sqrt{\alpha}x) \). The \( x \)-interval \((-\infty, \infty)\) goes into the half-axis, \([0, \infty)\). Now, (rewriting \( V \) for \( V^* \)),

\[
(3.3) \quad \int_t^\infty f(x) \, ds = \frac{\beta e^{2\sqrt{\alpha}x}}{2\sqrt{\alpha}} \int_x^\infty V(x) \exp(-2\sqrt{\alpha}s) \, ds
\]

\[
= \frac{\beta}{2\sqrt{\alpha}} \int_0^\infty e^{-2\sqrt{\alpha}\tau} V(x + \tau) \, d\tau = \beta \int_0^\infty e^{-2\sqrt{\alpha}\tau} \int_0^\tau V(x + s) \, ds \, d\tau
\]

\[
= \beta \int_0^\infty \tau e^{-2\sqrt{\alpha}\tau} \left[ \frac{1}{\tau} \int_0^\tau V(x + s) \, ds \right] \, d\tau
\]

\[
= \beta \int_0^\infty \tau e^{-2\sqrt{\alpha}\tau} \left[ \frac{1}{\tau} \int_0^{x+\tau} V(s) \, ds \right] \, d\tau.
\]

(note that both the first two integrals converge since \( V \) is bounded). Since \( M\{V(x)\} = 0 \) and \( V \) is a.p., then for every \( \varepsilon > 0 \) there exists \( \tau_0(\varepsilon) > 0 \) for which

\[
(3.4) \quad \sup_{x \in \mathbb{R}} \left| \frac{1}{\tau} \int_x^{x+\tau} V(s) \, ds \right| \leq \varepsilon
\]

for \( \tau \geq \tau_0 \) [2, p. 44]. Thus let \( T > 0 \) and rewrite (3.3) as an integral over \([0, T]\) plus an integral over \([T, \infty)\). Then

\[
(3.5) \quad \left| \int_0^\infty \tau e^{-2\sqrt{\alpha}\tau} \left[ \frac{1}{\tau} \int_x^{x+\tau} V(s) \, ds \right] \, d\tau \right| \leq M \int_0^T \tau e^{-2\sqrt{\alpha}\tau} \, d\tau
\]

as it is certainly the case that the integral appearing in the square parentheses is bounded, by \( M = M(T) \) say, as it is a continuous function of \( \tau \in [0, T] \). (Note that \( \sup\{M(T): T \geq 0\} < \infty \) on account of (3.4).) Moreover, since \( M\{V(x)\} = 0 \) we have

\[
(3.6) \quad \left| \int_T^\infty \tau e^{-2\sqrt{\alpha}\tau} \left[ \frac{1}{\tau} \int_x^{x+\tau} V(s) \, ds \right] \, d\tau \right| \leq \sup_{\tau \in [T, \infty)} \left| \frac{1}{\tau} \int_x^{x+\tau} V(s) \, ds \right| \cdot \int_T^\infty \tau e^{-2\sqrt{\alpha}\tau} \, d\tau
\]

\[
\leq \varepsilon(T) \int_T^\infty \tau e^{-2\sqrt{\alpha}\tau} \, d\tau.
\]

Combining the estimates (3.5), (3.7) and writing \( K = 2\sqrt{\alpha}T \) we obtain

\[
(3.8) \quad \left| t \int_t^\infty f(s) \, ds \right| \leq \frac{M|\beta|}{4\alpha} \left[ 1 - (K + 1)e^{-K} \right] + \frac{\varepsilon|\beta|}{4\alpha} (K + 1)e^{-K}
\]

\[
\leq \frac{|\beta|}{4\alpha} \left\{ \frac{M}{2} K^2 + \varepsilon(T) \right\}.
\]

We may now let \( T \to \infty \) in such a way that \( T = O(\alpha^{-1/2}) \) as \( \alpha \to 0^+ \). Then, the uniformity of the mean value (3.4) will imply that \( \varepsilon(t) \to 0 \) uniformly in \( x \) (see (3.6)). Moreover we will also have \( K \to 0 \) (as \( \alpha \to 0^+ \)).

Hence if \( \alpha > 0 \) is sufficiently small we see that

\[
(3.9) \quad \frac{|\beta|}{4\alpha} \left\{ \frac{M}{2} K^2 + \varepsilon(T) \right\} \leq \frac{1}{4}.
\]
i.e., if $|\beta| \leq \alpha \Psi(\alpha)$, where $\Psi(\alpha) = \{MK^2/2 + \varepsilon(T)\}^{-1}$ for an appropriately large $T$ which we then fix, then (3.2) will be nonoscillatory (and so disconjugate) on $[0, \infty)$ on account of [5, Theorem 6; 7], i.e., (3.2) will be disconjugate on $(-\infty, \infty)$. It follows from (3.10) that the disconjugacy domain just touches the $\beta$-axis at the origin, and at $(0,0)$ we have a vertical tangent!

We now return to (3.2). Assume $m > 0$. We set $\nu = 0$ in (3.1), i.e., $\alpha = -m \mu = -m \beta$ in (3.2). Then from the preceding discussion it follows that the line $\alpha + m \beta = 0$ must intersect the disconjugacy domain of (3.2) for some $\alpha > 0$ and some range of negative $\beta$'s, say, $0 > \beta \geq \beta_0$. Similarly if $m < 0$, we may find such a range of positive $\beta$'s, $0 < \beta \leq \beta_1$. In either case there exists $\mu \neq 0$ for which (3.1) (with $\nu = 0$) is disconjugate on $\mathbb{R}$. This completes the proof of the necessity and of the theorem.

REMARK. The proof of the necessity shows that the disconjugacy domain of an equation (3.2) with $V^*$ a.p. and $M\{V^*(x)\} = 0$ has a (boundary with a) vertical tangent at $(0,0)$ and lies completely in the right half-plane $\{\alpha > 0\} \cup \{(0,0)\}$. This extends a corresponding result of Markus and Moore [4, p. 106, Theorem 6] wherein it is further assumed that $v(x)$ (defined earlier) is also a.p. Furthermore the necessity merely required the uniformity of the mean-value of $V$ and consequently holds for potentials which may not be a.p. For example Stepanoff, Weyl/Besicovitch a.p. functions inherit this property as well as many other (nongeneralized a.p.) functions.

REFERENCES


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