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A NEW FIXED POINT THEOREM ON DEMI-COMPACT 1-SET-CONTRACTION MAPPINGS

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ABSTRACT. W. V. Petryshyn studied the fixed point theorem of demi-compact mappings. This paper gives a new fixed point theorem of demi-compact 1-set-contraction mappings in the norm form.

W. V. Petryshyn [**1**,**2**] studied the fixed point theorem of demi-compact mappings.

This paper gives a new fixed point theorem on demi-compact 1-set-contraction mappings. Let P be a cone of a real Banach space E. Set

$$P_r = \{x \in P \mid \|x\| < r\}, \qquad \partial P_r = \{x \in P \mid \|x\| = r\},$$
$$P_R = \{x \mid x \in P, \|x\| < R\}, \qquad \partial P_R = \{x \mid x \in P, \|x\| = R\},$$
$$P_{r,R} = \{x \in P \mid r < \|x\| < R\}, \qquad \overline{P}_{r,R} = \{x \in P \mid r \le \|x\| \le R\}, \qquad 0 < r < R.$$

First we state a result obtained by A. J. Potter [**3**], R. D. Nussbaum [**4**], and P. M. Fitzpatrick and W. V. Petryshyn [**5**].

LEMMA 1. *Let A: $\overline{P}_R \to P$ be a condensing mapping. Suppose that A satisfies*
(1) *There is some $w > \theta$ such that $x - Ax \neq \lambda w$ for any $\lambda > 0$ and $x \in \partial P_r$,*
(2) *$\lambda x \neq Ax$ for $\lambda > 1$ and $x \in \partial P_R$.*
Then A has a fixed point $x^* \in \overline{P}_{r,R}$.

REMARK. Lemma 1 is a particular condition of Theorem 3.2 of [**5**] and Remark 1.2 of [**4**].

COROLLARY [**4**]. *Let A: $\overline{P}_R \to P$ be condensing. Suppose that A satisfies one of the following conditions*:
(1)′ *$x \in \partial P_R \Rightarrow Ax \not\geq x$ and $x \in \partial P_r \Rightarrow Ax \not\leq x$,*
(2)′ *$x \in \partial P_r \Rightarrow Ax \not\geq x$ and $x \in \partial P_R \Rightarrow Ax \not\leq x$.*
Then A has a fixed point $x^* \in \overline{P}_{r,R}$.

Obviously, suppose that condition (1)′ is satisfied. Then
(i) $x \in \partial P_R$, $\lambda \ge 1 \Rightarrow Ax \neq \lambda x$ (if $Ax = \lambda x$ then $Ax = \lambda x \ge x$).
(ii) Let any $h > \theta$. Then $x \in \partial P_r$, $\lambda \ge 0 \Rightarrow x - Ax \neq \lambda h$ (if $x - Ax = \lambda h$, then $x - Ax \ge \theta$, i.e. $Ax \le x$). So A has a fixed point $x^* \in \overline{P}_{r,R}$.

Analogously, we can prove that A has a fixed point $x^* \in \overline{P}_{r,R}$ under (2)′.

LEMMA 2. *Let P be a cone of a real Banach space E, and let the norm $\|x\|$ be increasing with respect to P. Suppose that A: $\overline{P}_R \to P$ is a k-set-contraction mapping $(0 < k < 1)$, which satisfies one of the following conditions*:

$$(\mathrm{H}_1) \qquad x \in \partial P_r \Rightarrow \|Ax\| \le \|x\|; \qquad x \in \partial P_R \Rightarrow \|Ax\| \ge \|x\|,$$

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or

$$(\mathrm{H}_2) \qquad x \in \partial P_R \Rightarrow \|Ax\| \le \|x\|; \qquad x \in \partial P_r \Rightarrow \|Ax\| \ge \|x\|.$$

Then A has a fixed point in $\overline{P}_{r,R}$.

PROOF. We prove only that Lemma 1 holds under (H_1). A is a strict-set-contraction.

We set the operator as follows:

$$\tilde{A}_n x = \begin{cases} \left(1 + \dfrac{\|x\| - s}{n(R-s)}\right) Ax, & x \in P, s \le \|x\| \le R, \\ \left(1 - \dfrac{s - \|x\|}{n(s-r)}\right) Ax, & x \in P, r \le \|x\| < s, \end{cases}$$

where $s = \frac{1}{2}(r+R)$. $\tilde{A}_n$ is a continuous and bounded operator.

We discuss the operator

$$B_n x = (\|x\|/nK)Ax, \qquad x \in \overline{P}_{r,R}, K = \min\{R-s, s-r\}.$$

Let Ω be a open set in $\overline{P}_{r,R}$, by definition and property of the measure $\alpha(\Omega)$ of noncompactness of Ω,

$$\alpha(\Omega) = \inf\left\{ d | s_i \subset \Omega, \bigcup_{i=1}^{N} s_i = \Omega, \operatorname{diam} s_i < d \right\},$$

$$\alpha(\Omega) \le \alpha(\overline{P}_{r,R}), \quad B_n(s_i) \subset B_n(\Omega), \quad \bigcup_{i=1}^{N} B_n(s_i) = B_n(\Omega),$$

$$\operatorname{diam} B_n(s_i) \le (2R/nK) \operatorname{diam} A(s_i),$$

$$\alpha(B_n(\Omega)) \le (2R/nK)\alpha(A(\Omega)) \le (2R/nK)k\alpha(\Omega).$$

Hence B_n is a $(2R/nK)k$-set-contraction mapping, as $n \to \infty$, $(2R/nK)k \to 0$. So when n is large enough, B_n is a strict-set-contraction mapping on $\overline{P}_{r,R}$.

We set operators

$$A_n^{(1)} x = \left(1 + \frac{\|x\| - s}{n(R-s)}\right) Ax, \qquad x \in P, s \le \|x\| \le R,$$

$$A_n^{(2)} x = \left(1 - \frac{s - \|x\|}{n(s-r)}\right) Ax, \qquad x \in P, r \le \|x\| < s.$$

Obviously $A_n^{(1)}$ and $A_n^{(2)}$ are $\tilde{k}$-set-contraction mappings, where

$$\tilde{k} = \left(1 + \frac{2R+s}{nK}\right) k.$$

When n is large enough, $0 < \tilde{k} < 1$. Hence $A_n^{(1)}$ and $A_n^{(2)}$ are strict-set-contraction mappings for large enough n.

Set

$$P_1 = \{x | x \in P, s \le \|x\| \le R\},$$

$$P_2 = \{x | x \in P, r \le \|x\| < s\}, \qquad \overline{P}_{r,R} = P_1 \cup P_2.$$

Hence,

$$\begin{aligned}
\Omega &= (\Omega \cap P_1) \cup (\Omega \cap P_2),\\
\tilde{A}_n(\Omega) &= [\tilde{A}_n(\Omega \cap P_1)] \cup [\tilde{A}_n(\Omega \cap P_2)]\\
&= [A_n^{(1)}(\Omega \cap P_1)] \cup [A_n^{(2)}(\Omega \cap P_2)],\\
\alpha(A_n(\Omega)) &= \alpha\{[A_n^{(1)}(\Omega \cap P_1)] \cup [A_n^{(2)}(\Omega \cap P_2)]\}\\
&= \operatorname{Max}\{\alpha[A_n^{(1)}(\Omega \cap P_1)], \alpha[A_n^{(2)}(\Omega \cap P_2)]\}\\
&\le \tilde{k}\alpha(\Omega), \qquad 0 < \tilde{k} < 1.
\end{aligned}$$

Then $\tilde{A}_n$ is a strict-set-contraction mapping. When n is large enough, set

$$A_n x = \begin{cases} \tilde{A}_n x, & r \le \|x\| \le R,\\ Ax, & 0 \le \|x\| < r. \end{cases}$$

A_n satisfies a condition of the Corollary of Lemma 1.

In fact, if there is an $x_0 \in \partial P_r$ such that $A_n x_0 \ge x_0$, since norm is monotonic increasing and, by condition (H_1),

$$r = \|x_0\| \le \|A_n x_0\| = (1 - 1/n)\|Ax_0\| \le (1 - 1/n)\|x_0\| = (1 - 1/n)r < r.$$

This is contradiction. Then $x \in \partial P_r$, $A_n x \not\ge x$.

If there is an $x_1 \in \partial P_R$ such that $A_n x_1 \le x_1$, we obtain

$$R = \|x_1\| \ge \|A_n x_1\| = (1 + 1/n)\|Ax_1\| \ge (1 + 1/n)\|x_1\| = (1 + 1/n)R > R.$$

This is also contradiction. Then $x \in \partial P_R$, $A_n x \not\le x$.

Hence A_n satisfies a condition of the Corollary of Lemma 1, and there exists $x_n^* \in \overline{P}_{r,R}$ such that $A_n x_n^* = x_n^*$.

If P_1 includes, without loss of generality, subsequence $\{x_{n_k}^*\}$ of $\{x_n^*\}$,

$$x_{n_k}^* = A_{n_k} x_{n_k}^* = \left(1 + \frac{\|x_{n_k}^*\| - s}{n_k(R - s)}\right) Ax_{n_k}^*. \tag{1}$$

Since A is a strict-set-contraction mapping, the set $\{\|Ax_{n_k}^*\|\}$ must be bounded.

$$\|x_{n_k}^* - Ax_{n_k}^*\| = \frac{\|x_{n_k}^*\| - s}{n_k(R - s)}\|Ax_{n_k}^*\| < \frac{1}{n_k}\|Ax_{n_k}^*\| \to 0 \qquad (n_k \to \infty),$$

i.e., $x_{n_k}^* - Ax_{n_k}^* \to \theta$ $(n_k \to \infty)$.

Since a strict-set-contraction mapping is a demi-compact 1-set-contraction, then there exists a convergent subsequence of $\{x_{n_k}^*\}$, which we write down also as $\{x_{n_k}^*\}$, and $x_{n_k}^* \to x^*$ $(n_k \to \infty)$. Since $\overline{P}_{r,R}$ is closed, then $x^* \in \overline{P}_{r,R}$. By (1), as $n_k \to \infty$, we imply that $Ax^* = x^*$. This proves that A has a fixed point $x^* \in \overline{P}_{r,R}$ under condition (H_1).

Analogously, we can prove that A has a fixed point $x^* \in \overline{P}_{r,R}$ under condition (H_2). Q.E.D.

We obtain the following main theorem.

THEOREM. *Let P be a cone of a real Banach space E, and let norm be increasing with respect to P. Suppose that A: $\overline{P}_R \to P$ demi-compact 1-set-contraction mapping and there is a $\delta > 0$ such that*

(i) $x \in \partial P_r \Rightarrow \|Ax\| \le \|x\|$, $x \in \partial P_R \Rightarrow \|Ax\| \ge (1 + \delta)\|x\|$,

or

(ii) $x \in \partial P_R \Rightarrow \|Ax\| \le \|x\|,\ x \in \partial P_r \Rightarrow \|Ax\| \ge (1+\delta)\|x\|$.

Then A has a fixed point in $\overline{P}_{r,R}$.

PROOF. Since A: $\overline{P}_R \to P$ is a 1-set-contraction mapping, for an arbitrary open subset $\Omega \subset \overline{P}_{r,R}$, $\alpha(A(\Omega)) \le \alpha(\Omega)$.

We construct a new operator as follows:

$$A_n x = \lambda_n A x, \qquad \lambda_n = (n-1)/n.$$

Hence,

$$\alpha(A_n(\Omega)) = \alpha(\lambda_n A(\Omega)) = \lambda_n \alpha(A(\Omega)) \le \lambda_n \alpha(\Omega).$$

Then A_n is a strict-set-contraction mapping.

If condition (i) holds when n is large enough such that $1 > \lambda_n > 1/(1+\delta)$, then

$$\begin{aligned} x \in \partial P_r, &\qquad \|A_n x\| = \lambda_n \|Ax\| \le \|x\|, \\ x \in \partial P_R, &\qquad \|A_n x\| = \lambda_n \|Ax\| \ge \|Ax\|/(1+\delta) \ge \|x\|. \end{aligned}$$

Hence A_n satisfies a condition of Lemma 2, and there exist $x_n \in \overline{P}_{r,R}$ such that $A_n x_n = x_n$, i.e. $\lambda_n A x_n = x_n$,

$$\begin{aligned} x_n - Ax_n &= x_n - \lambda_n A x_n + \lambda_n A x_n - A x_n \\ &= (\lambda_n - 1) A x_n \to \theta \qquad (n \to \infty). \end{aligned}$$

Because A is a demi-compact mapping, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0 \in \overline{P}_{r,R}$. By continuity of A, $x_0 = Ax_0$.

Analogously, we can prove that A has a fixed point $x^* \in \overline{P}_{r,R}$ under condition (ii).

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