A NEW FIXED POINT THEOREM ON DEMI-COMPACT 1-SET-CONTRACTION MAPPINGS

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ABSTRACT. W. V. Petryshyn studied the fixed point theorem of demi-compact mappings. This paper gives a new fixed point theorem of demi-compact 1-set-contraction mappings in the norm form.

W. V. Petryshyn [1,2] studied the fixed point theorem of demi-compact mappings.

This paper gives a new fixed point theorem on demi-compact 1-set-contraction mappings. Let P be a cone of a real Banach space E. Set

\[ P_r = \{ x \in P \mid \|x\| < r \}, \quad \partial P_r = \{ x \in P \mid \|x\| = r \}, \]
\[ P_R = \{ x \in P \mid \|x\| < R \}, \quad \partial P_R = \{ x \in P \mid \|x\| = R \}, \]
\[ P_{r,R} = \{ x \in P \mid r < \|x\| < R \}, \quad P_{r,R} = \{ x \in P \mid r \leq \|x\| \leq R \}, \quad 0 < r < R. \]

First we state a result obtained by A. J. Potter [3], R. D. Nussbaum [4], and P. M. Fitzpatrick and W. V. Petryshyn [5].

**Lemma 1.** Let A: \( P_R \to P \) be a condensing mapping. Suppose that A satisfies

1. There is some \( w > 0 \) such that \( x - Ax \nless w \) for any \( X > 0 \) and \( x \in \partial P_r \),
2. \( Ax \nless Xx \) for \( X > 1 \) and \( x \in \partial P_R \).

Then A has a fixed point \( x^* \in P_{r,R} \).

**Remark.** Lemma 1 is a particular condition of Theorem 3.2 of [5] and Remark 1.2 of [4].

**Corollary [4].** Let A: \( P_R \to P \) be condensing. Suppose that A satisfies one of the following conditions:

1(1)' \( x \in \partial P_r \Rightarrow Ax \nless x \) and \( x \in \partial P_r \Rightarrow Ax \nless x \),
2(2)' \( x \in \partial P_r \Rightarrow Ax \nless x \) and \( x \in \partial P_R \Rightarrow Ax \nless x \).

Then A has a fixed point \( x^* \in P_{r,R} \).

Obviously, suppose that condition (1)\( ' \) is satisfied. Then

(i) \( x \in \partial P_R, \lambda \geq 1 \Rightarrow Ax \nless Xx \) (if \( Ax = \lambda x \) then \( Ax = \lambda x \).

(ii) Let any \( h > 0 \). Then \( x \in \partial P_r, \lambda \geq 0 \Rightarrow x - Ax \nless \lambda h \) (if \( x - Ax = \lambda h \), then \( x - Ax \geq h \), i.e. \( Ax \leq x \)). So A has a fixed point \( x^* \in P_{r,R} \).

Analogously, we can prove that A has a fixed point \( x^* \in P_{r,R} \) under (2)\( ' \).

**Lemma 2.** Let P be a cone of a real Banach space E, and let the norm \( \|x\| \) be increasing with respect to P. Suppose that A: \( P_R \to P \) is a k-set-contraction mapping \( 0 < k < 1 \), which satisfies one of the following conditions:

\[ (H_1) \quad x \in \partial P_r \Rightarrow \|Ax\| \leq \|x\|; \quad x \in \partial P_R \Rightarrow \|Ax\| \geq \|x\|, \]

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or

\[ (H_2) \quad x \in \partial P_R \Rightarrow \|Ax\| \leq \|x\|; \quad x \in \partial P_r \Rightarrow \|Ax\| \geq \|x\|. \]

Then \( A \) has a fixed point in \( \overline{P}_{r,R} \).

**Proof.** We prove only that Lemma 1 holds under \((H_1)\). \( A \) is a strict-set-contraction.

We set the operator as follows:

\[
\tilde{A}_n x = \begin{cases} 
\left( 1 + \frac{\|x\| - s}{n(R - s)} \right) Ax, & x \in P, s \leq \|x\| \leq R, \\
\left( 1 - \frac{s - \|x\|}{n(s - r)} \right) Ax, & x \in P, r \leq \|x\| < s,
\end{cases}
\]

where \( s = \frac{1}{2}(r + R) \). \( \tilde{A}_n \) is a continuous and bounded operator.

We discuss the operator

\[
B_n x = \left( \frac{\|x\|}{nK} \right) Ax, \quad x \in \overline{P}_{r,R}, K = \min\{R - s, s - r\}.
\]

Let \( \Omega \) be an open set in \( \overline{P}_{r,R} \), by definition and property of the measure \( \alpha(\Omega) \) of noncompactness of \( \Omega \),

\[
\alpha(\Omega) = \inf \left\{ d | s_i \subset \Omega, \bigcup_{i=1}^N s_i = \Omega, \text{diam } s_i < d \right\},
\]

\[
\alpha(\Omega) \leq \alpha(\overline{P}_{r,R}), \quad B_n(s_i) \subset B_n(\Omega), \quad \bigcup_{i=1}^N B_n(s_i) = B_n(\Omega),
\]

\[
\text{diam } B_n(s_i) \leq (2R/nK) \text{diam } A(s_i), \quad \alpha(B_n(\Omega)) \leq \frac{(2R/nK)\alpha(A(\Omega))}{(2R/nK)\alpha(\Omega)} \leq (2R/nK)\alpha(\Omega).
\]

Hence \( B_n \) is a \((2R/nK)\kappa\)-set-contraction mapping, as \( n \to \infty \), \((2R/nK)\kappa \to 0\). So when \( n \) is large enough, \( B_n \) is a strict-set-contraction mapping on \( \overline{P}_{r,R} \).

We set operators

\[
A_n^{(1)} x = \left( 1 + \frac{\|x\| - s}{n(R - s)} \right) Ax, \quad x \in P, s \leq \|x\| \leq R,
\]

\[
A_n^{(2)} x = \left( 1 - \frac{s - \|x\|}{n(s - r)} \right) Ax, \quad x \in P, r \leq \|x\| < s.
\]

Obviously \( A_n^{(1)} \) and \( A_n^{(2)} \) are \( \kappa \)-set-contraction mappings, where

\[
\tilde{k} = \left( 1 + \frac{2R + s}{nK} \right) k.
\]

When \( n \) is large enough, \( 0 < \tilde{k} < 1 \). Hence \( A_n^{(1)} \) and \( A_n^{(2)} \) are strict-set-contraction mappings for large enough \( n \).

Set

\[
P_1 = \{ x | x \in P, s \leq \|x\| \leq R \}, \quad P_2 = \{ x | x \in P, r \leq \|x\| < s \}, \quad \overline{P}_{r,R} = P_1 \cup P_2.
\]
Hence,
\[ \Omega = (\Omega \cap P_1) \cup (\Omega \cap P_2), \]
\[ \tilde{A}_n(\Omega) = [\tilde{A}_n(\Omega \cap P_1)] \cup [\tilde{A}_n(\Omega \cap P_2)] \]
\[ = [A_n^{(1)}(\Omega \cap P_1)] \cup [A_n^{(2)}(\Omega \cap P_2)], \]
\[ \alpha(A_n(\Omega)) = \alpha([A_n^{(1)}(\Omega \cap P_1)] \cup [A_n^{(2)}(\Omega \cap P_2)]) \]
\[ = \max\{\alpha(A_n^{(1)}(\Omega \cap P_1)), \alpha(A_n^{(2)}(\Omega \cap P_2))\} \]
\[ \leq \hat{k} \alpha(\Omega), \quad 0 < \hat{k} < 1. \]

Then \( \tilde{A}_n \) is a strict-set-contraction mapping. When \( n \) is large enough, set
\[ A_n x = \begin{cases} \tilde{A}_n x, & r \leq ||x|| \leq R, \\ A x, & 0 \leq ||x|| < r. \end{cases} \]

\( A_n \) satisfies a condition of the Corollary of Lemma 1.

In fact, if there is an \( x_0 \in \partial P_r \) such that \( A_n x_0 \geq x_0 \), since norm is monotonic increasing and, by condition \( (H_1) \),
\[ r = ||x_0|| \leq ||A_n x_0|| = (1 - 1/n)||Ax_0|| \leq (1 - 1/n)||x_0|| = (1 - 1/n)r < r. \]

This is contradiction. Then \( x \in \partial P_r, A_n x \geq x \).

If there is an \( x_1 \in \partial P_r \) such that \( A_n x_1 \leq x_1 \), we obtain
\[ R = ||x_1|| \geq ||A_n x_1|| = (1 + 1/n)||Ax_1|| \geq (1 + 1/n)||x_1|| = (1 + 1/n)R > R. \]

This is also contradiction. Then \( x \in \partial P_r, A_n x \leq x \).

Hence \( A_n \) satisfies a condition of the Corollary of Lemma 1, and there exists \( x_{n_k}^* \in \overline{P}_{r,R} \) such that \( A_n x_{n_k}^* = x_{n_k}^* \).

If \( P_1 \) includes, without loss of generality, subsequence \( \{x_{n_k}^*\} \) of \( \{x_n^*\} \),
\[ x_{n_k}^* = A_n x_{n_k}^* = \left(1 + \frac{||x_{n_k}^*|| - s}{n_k (R - s)}\right) Ax_{n_k}^*. \]

Since \( A \) is a strict-set-contraction mapping, the set \( \{||Ax_{n_k}^*||\} \) must be bounded.
\[ ||x_{n_k}^* - Ax_{n_k}^*|| = \frac{||x_{n_k}^*|| - s}{n_k (R - s)} ||Ax_{n_k}^*|| < \frac{1}{n_k} ||Ax_{n_k}^*|| \rightarrow 0 \quad (n_k \rightarrow \infty), \]
i.e., \( x_{n_k}^* - Ax_{n_k}^* \rightarrow 0 \ (n_k \rightarrow \infty) \).

Since a strict-set-contraction mapping is a demi-compact 1-set-contraction, then there exists a convergent subsequence of \( \{x_{n_k}^*\} \), which we write down also as \( \{x_{n_k}^*\} \), and \( x_{n_k}^* \rightarrow x^* \ (n_k \rightarrow \infty) \). Since \( \overline{P}_{r,R} \) is closed, then \( x^* \in \overline{P}_{r,R} \). By (1), as \( n_k \rightarrow \infty \), we imply that \( Ax^* = x^* \). This proves that \( A \) has a fixed point \( x^* \in \overline{P}_{r,R} \) under condition \( (H_1) \).

Analogously, we can prove that \( A \) has a fixed point \( x^* \in \overline{P}_{r,R} \) under condition \( (H_2) \). Q.E.D.

We obtain the following main theorem.

**Theorem.** Let \( P \) be a cone of a real Banach space \( E \), and let norm be increasing with respect to \( P \). Suppose that \( A: \overline{P}_R \rightarrow P \) demi-compact 1-set-contraction mapping and there is a \( \delta > 0 \) such that
\[ (i) \quad x \in \partial P_r \Rightarrow ||Ax|| \leq ||x||, \quad x \in \partial P_R \Rightarrow ||Ax|| \geq (1 + \delta)||x||, \]
or

(ii) \( x \in \partial P_R \Rightarrow \|Ax\| \leq \|x\|, \ x \in \partial P_r \Rightarrow \|Ax\| \geq (1 + \delta)\|x\| \).

Then \( A \) has a fixed point in \( \overline{P}_{r,R} \).

**Proof.** Since \( A: \overline{P}_R \rightarrow P \) is a 1-set-contraction mapping, for an arbitrary open subset \( \Omega \subset \overline{P}_{r,R}, \ \alpha(A(\Omega)) \leq \alpha(\Omega) \).

We construct a new operator as follows:

\[
A_n x = \lambda_n Ax, \quad \lambda_n = (n - 1)/n.
\]

Hence,

\[
\alpha(A_n(\Omega)) = \alpha(\lambda_n A(\Omega)) = \lambda_n \alpha(A(\Omega)) \leq \lambda_n \alpha(\Omega).
\]

Then \( A_n \) is a strict-set-contraction mapping.

If condition (i) holds when \( n \) is large enough such that \( 1 > \lambda_n > 1/(1 + \delta) \), then

\[
x \in \partial P_r, \quad \|A_n x\| = \lambda_n \|Ax\| \leq \|x\|, \\
x \in \partial P_R, \quad \|A_n x\| = \lambda_n \|Ax\| \geq \|Ax\|/(1 + \delta) \geq \|x\|.
\]

Hence \( A_n \) satisfies a condition of Lemma 2, and there exist \( x_n \in \overline{P}_{r,R} \) such that \( A_n x_n = x_n \), i.e. \( \lambda_n Ax_n = x_n \),

\[
x_n - Ax_n = x_n - \lambda_n Ax_n + \lambda_n Ax_n - Ax_n \\
= \lambda_n - 1)Ax_n \rightarrow 0 \quad (n \rightarrow \infty).
\]

Because \( A \) is a demi-compact mapping, there exists a convergent subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \rightarrow x_0 \in \overline{P}_{r,R} \). By continuity of \( A \), \( x_0 = Ax_0 \).

Analogously, we can prove that \( A \) has a fixed point \( x^* \in \overline{P}_{r,R} \) under condition (ii).

**References**


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