PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 2, June 1986

# A POINCARÉ-TYPE INEQUALITY FOR SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS

WILLIAM P. ZIEMER[1]

ABSTRACT. A sharpened version of the Poincaré inequality is shown to hold for solutions of a large class of second order elliptic equations.

**1. Introduction.** If $\Omega$ is a sufficiently regular domain in $R^n$ and $u$ a smooth function defined on $\Omega$, then a well-known form of the Poincaré inequality states that there is a constant $C$ such that

$$\|u - \bar{u}(\Omega)\|_p \leq C\|\nabla u\|_p \tag{1}$$

where $1 \leq p < \infty$ and $\bar{u}(\Omega)$ denotes the integral average of $u$ over $\Omega$. If $u$ has compact support in $\Omega$ or vanishes on a set of positive measure, then (1) holds with $\bar{u}(\Omega)$ replaced by 0. Inequalities of this type have many applications in analysis, especially in differential equations, and they admit many variations and generalizations, cf. [**ME**, **MZ**, **P**, **Z**].

An interesting result similar to (1) was established in [**AS**] for holomorphic (harmonic) functions $u$ defined on a star-shaped domain in $\mathbf{C}$ where $\bar{u}(\Omega)$ is replaced by the value of $u$ at any distinguished point in $\Omega$. In [**JS**] a similar result was shown to hold for solutions of second order elliptic equations with smooth coefficients defined on an open ball. The purpose of this note is to give a rather comprehensive treatment of this type inequality. Our results are valid for solutions to a large class of nonlinear equations, including those with measurable coefficients, defined on a bounded Lipschitz domain $\Omega$. In the case of linear elliptic equations in divergence form whose principal coefficients are Hölder continuous, by suitably modifying a method in [**AS**], it is shown that $\Omega$ can be taken as any bounded domain whose boundary can be expressed locally as the graph of a continuous function.

**2. Equations with measurable coefficients.** In this section we consider a bounded, connected Lipschitz domain $\Omega$ in $R^n$. Let $u$ be an element of the Sobolev space $W^{1,p}(\Omega)$ $(1 < p < n)$ that is a weak solution of the equation

$$\operatorname{div} A(x \cdot u, \nabla u) = B(x, u, \nabla u) \tag{2}$$

where $A$ and $B$ are Borel measurable vector-valued and scalar-valued functions, respectively, defined on $\Omega \times R^1 \times R^n$. The function $u$ is said to be a weak solution of (2) if

$$\int A \cdot \nabla\varphi + B\varphi = 0$$

Received by the editors June 10, 1985.
1980 *Mathematics Subject Classification*. Primary 35J15; Secondary 46E35.
*Key words and phrases*. Poincaré inequality, elliptic equations.
[1]Research supported in part by a grant from the National Science Foundation.

whenever $\varphi \in W_0^{1,p}(\Omega)$. The functions $A$ and $B$ are required to satisfy the following structural inequalities:

$$
\begin{aligned}
|A(x,z,w)| &\le a_0|w|^{p-1} + a_1|z|^{p-1} + a_2,\\
|B(x,z,w,)| &\le b_1|w|^{p-1} + b_2|z|^{p-1} + b_3,\\
A(x,z,w)\cdot w &\ge |w|^p - c_1|z|^p - c_2.
\end{aligned} \tag{3}
$$

Here, $a_0$ is a positive constant and the remaining coefficients are all elements of suitable $L^p(\Omega)$ spaces. The reader may consult [**S** and **T**] for a general development of the subject. We will only need that weak solutions $u$ of (2) are locally Hölder continuous ($C^{0,\alpha}$), a fact that follows from the Harnack inequality. The constant $\alpha$ depends only on the structure (3). In particular, $u$ satisfies the following weak Harnack inequality: Let $B(3r)$ denote the ball of radius $3r$ and assume $B(3r) \subset \Omega$. Then

$$
\sup_{B(r)} |u| \le C \left[ \fint_{B(2r)} |u|^p \right]^{1/p} + K \tag{4}
$$

where $K$ depends only on $r$ and the structural inequalities (3). Here, $\fint$ denotes the integral average. It has been shown in [**DBT**] that (4) remains valid if $p$ is replaced by any $q > 0$. We denote the $L^p$ norm of $u$ on $\Omega$ by $\|u\|_{p;\Omega}$.

2.1 THEOREM. *Let $\Omega \subset R^n$ be a bounded connected Lipschitz domain and let $u \in W^{1,p}(\Omega)$ be a weak solution of* (2). *For each $x_0 \in \Omega$ there is a constant $C$ that depends only on the structure* (3), $\|u\|_{p;\Omega}$, *and $\Omega$ such that*

$$
\|u - u(x_0)\|_{p;\Omega} \le C\|\nabla u\|_{p;\Omega}. \tag{5}
$$

PROOF. Assume to the contrary that for every positive integer $j$ there is a weak solution $u_j$ such that

$$
\|u_j - u_j(x_0)\|_{p;\Omega} > j\|\nabla u_j\|_{p;\Omega}. \tag{6}
$$

Although $u_j - u_j(x_0)$ is not necessarily a solution of (2), it is a solution of an associated equation with structure similar to that of (3). For example, if a solution $u$ is replaced by $\bar{u} = u - c$, then $\bar{u}$ satisfies the equation

$$
\operatorname{div} \overline{A}(x,\bar{u},\nabla\bar{u}) = \overline{B}(x,\bar{u},\nabla\bar{u}) \tag{7}
$$

with $\overline{A}(x,\bar{u},\nabla\bar{u}) = A(x,\bar{u}+c,\nabla\bar{u}+c)$ and a similar formula for $\overline{B}$. Clearly the structure for (7) is the same as (3) except for the introduction of $c$ in the coefficients. In our situation, $c = u_j(x_0)$ and $c$ is dominated by an expression involving $\|u_j\|_{p;\Omega}$ by (4). Therefore, by replacing $u_j - u_j(x_0)$ by $u_j$, we may assume that $u_j(x_0) = 0$. Similarly, we may replace $u_j$ by $u_j/\|u_j\|_{p;\Omega}$ and thus assume that $\|u_j\|_{p;\Omega} = 1$. Now we have

$$
\|u_j\|_{p;\Omega} > j\|\nabla u_j\|_{p;\Omega} \tag{8}
$$

with $\|u_j\|_{p;\Omega} = 1$. Therefore, since the Sobolev norms of the $u_j$ are bounded, there exist $u \in W^{1,p}(\Omega)$ and a subsequence of $u_j$ (which we still denote by $u_j$) such that $u_j \to u$ weakly in $W^{1,p}(\Omega)$. Because $\Omega$ is assumed to be Lipschitz, it follows from an extension of the Kondrachov compactness theorem, cf. [**GT**, §7.12], that

$$
\|u_j - u\|_{p;\Omega} \to 0.
$$

It follows from (8) that $\|\nabla u_j\|_{p;\Omega} \to 0$ and therefore that $\|u_j\|_{p;\Omega} \to 1$. Consequently, $\|u\|_{p;\Omega} = 1$ and $\|\nabla u\|_{p;\Omega} = 0$. Recall that $u_j$ is Hölder continuous on each compact subset of $\Omega$, and by the weak Harnack inequality (4) that the sequence $\{u_j\}$ is uniformly bounded. Therefore, by Ascoli's theorem, a subsequence converges uniformly to $u$ on each compact subset of $\Omega$. Thus, $u$ is continuous on $\Omega$ and $u(x_0) = 0$. Hence $u \equiv 0$ on $\Omega$ because $\|\nabla u\|_{p;\Omega} = 0$. This contradicts the fact that $\|u\|_{p;\Omega} = 1$. $\square$

2.2 REMARK. The dependence of the constant $C$ on $\|u\|_{p;\Omega}$ can be avoided if $A$ and $B$ are both homogeneous of the same degree in $z$ and $w$. In this case $u_j/\|\nabla u_j\|_{p;\Omega}$ is again a solution and therefore, in the proof above, we may assume that $\|\nabla u_j\|_{p;\Omega} = 1$. Since $\Omega$ is a bounded Lipschitz domain, each $u_j$ can be extended to all of $R^n$ with an equivalent Sobolev norm. Multiplying each $u_k$ by a smooth function with compact support that is identically 1 on the closure of $\Omega$ and applying the classical Poincaré inequality implies that the norms $\|u_j\|_{2;\Omega}$ remain uniformly bounded. Hence, the above proof still applies. For example, the equation

$$D_i(|\nabla u|^{p-2} a^{ij}(x) D_j u) = |\nabla u|^{p-1}$$

is acceptable where $a^{ij} \in L^\infty(\Omega)$ and $[a^{ij}]$ is positive definite. In the next section we will see that this dependence can also be avoided if the equation is linear with Hölder continuous coefficients.

**3. Linear equations.** We will now consider linear equations of the form

$$Lu = D_i(a^{ij}(x) D_j u + b^i(x) u) + c^i(x) D_i u + d(x) u \tag{9}$$

whose coefficients are initially assumed to be bounded measurable functions on a bounded, connected domain $\Omega \subset R^n$. In addition, it will be assumed that $L$ is strictly elliptic in $\Omega$; that is, there exists a positive number $\lambda$ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall x \in \Omega, \xi \in R^n. \tag{10}$$

These equations are, of course, a subclass of those considered in §2.

3.1 DEFINITION. For fixed $x_0 \in \Omega$, let

$$D_2(\Omega) = W^{1,2}_{\text{loc}}(\Omega) \cap \{u\colon\ Lu = 0, u(x_0) = 0, \|\nabla u\|_{2;\Omega} < \infty\}. \tag{11}$$

3.2 LEMMA. *$D_2(\Omega)$ is a Banach space.*

PROOF. Let $\{u_i\}$ be a sequence in $D_2(\Omega)$ such that $\|\nabla u_i - \nabla u_j\|_{2;\Omega} \to 0$. Let $\Omega' \subset\subset \Omega$ be a domain with $x_0 \in \Omega'$ and let $\eta$ be a smooth function that is identically 1 on the closure of $\Omega'$ with spt $\eta \subset \Omega$. From the classical Poincaré inequality, it follows that the norms $\|\eta u_i\|_{2;\Omega}$ are uniformly bounded. Hence, from the weak Harnack inequality (4) we have that the solutions $u_i$ are uniformly bounded on $\Omega'$. Because they are Hölder continuous, Ascoli's theorem implies there is $u \in W^{1,p}(\Omega')$ such that $u_i \to u$ uniformly. Thus, $u(x_0) = 0$. Elementary considerations show that $u \in W^{1,p}_{\text{loc}}(\Omega)$ and $Lu = 0$. $\square$

A Poincaré-type inequality for linear equations with measurable coefficients is a consequence of the following result:

3.3 THEOREM. *If $D_2(\Omega) \subset L^2(\Omega)$, then the inclusion map is continuous.*

PROOF. We will employ the closed graph theorem to establish the result. To that end, let $u_i \to u$ in $D_2(\Omega)$ and let $u_i \to v$ in $L^2(\Omega)$. We will show that $u = v$.

Note that $D_j u_i \to D_j v$ in $\mathcal{D}'$ (in the sense of distributions) for any partial derivative $D_j$ because $u_i \to v$ in $\mathcal{D}'$. However, $D_j u_i \to D_j u$ and, therefore, $\nabla u = \nabla v$. This implies that $\nabla v \in L^2(\Omega)$ and $Lv = 0$. Arguing as in 3.2, we have that $\{u_i\}$ is uniformly bounded on compact subsets of $\Omega$ and locally Hölder continuous. An application of Ascoli's theorem implies that $u_i \to v$ uniformly on compact subsets and hence, that $v(x_0) = 0$. Because $u(x_0) = 0$ and $\nabla u = \nabla v$, it follows that $u = v$. The closed graph theorem now applies and the inclusion map is continuous. □

In view of Theorem 3, it is important to determine those domains $\Omega$ such that $D_2(\Omega) \subset L^2(\Omega)$. For this purpose we make the following definition.

3.4 DEFINITION. If $\Omega$ is a bounded domain, we will say that $\partial\Omega$ is locally a graph if, for each $x_0 \in \partial\Omega$, there is a neighborhood about $x_0$ which is the image, under a rigid motion, of a domain of the form

$$\{(x', y)\colon\ x' \in B^{n-1}(r), 0 < y < f(x')\} \tag{12}$$

where $B^{n-1}(r)$ is an open ball of radius $r$ in $R^{n-1}$ and $f\colon\ B^{n-1}(r) \to R^1$ is continuous and bounded away from 0.

Let us now assume that the coefficients of $L$ satisfy the following: $a^{ij}, b^i \in C^{0,\alpha}(\Omega), c^i, d \in L^\infty(\Omega)$.

3.5 THEOREM. *If $\Omega$ is a domain whose boundary $\partial\Omega$ is locally a graph and $L$ is as above, then $D_2(\Omega) \subset L^2(\Omega)$.*

PROOF. Because of the assumption on the coefficients of $L$, it follows that if $u \in D^2(\Omega)$, then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, cf. [**GT**, §8.36]. Let $x_0 \in \partial\Omega$ and consider a neighborhood $U$ of $x_0$ as described in Definition 3.4. For simplicity, assume that $U$ is of the form (12). For each $x \in U$, let $x' \in B^{n-1}(r)$ be such that $(x', y) = x$. Then

$$\begin{aligned} u(x) - u(x') &= \int_{x'}^{x} \nabla u(w) \cdot \frac{(x - x')}{|x - x'|}\, dw \\ &\le |x - x'| \int_0^1 |\nabla u(tx + (1-t)x')|\, dt. \end{aligned}$$

Hence

$$|u(x) - u(x')|^2 \le C(\Omega) \int_0^1 |\nabla u(tx + (1-t)x')|^2\, dt.$$

Note that

$$\int_U |u(x) - u(x')|^2\, dx \le C(\Omega) \int_0^1 t^{-1} \int_{tU} |\nabla u|^2\, dx\, dt.$$

Clearly, closure $sU \subset U$ for some $0 < s < 1$ and therefore $|\nabla u|^2 < k$ on $sU$. Hence, we have

$$\int_0^s t^{-1} \int_{tU} |\nabla u|^2\, dx\, dt < k|U|$$

where $|U|$ denotes the Lebesgue measure of $U$. Also,

$$\int_s^1 t^{-1} \int_{tU} |\nabla u|^2\, dx\, dt \le \int_s^1 t^{-1} \int_U |\nabla u|^2\, dx\, dt < \infty.$$

Thus,

$$\int_U |u(x) - u(x')|^2\, dx < \infty.$$

Since $u \in C^{1,\alpha}(\Omega)$, $u$ is bounded on $B^{n-1}(r)$. This implies that $\int_U |u(x)|^2\,dx < \infty$, and because $\partial\Omega$ can be covered by a finite number of such neighborhoods $U$, it follows that $\|u\|_{2;\Omega} < \infty$. $\square$

3.6 REMARK. Of course, Lipschitz domains satisfy Definition 3.4. Likewise, star-shaped domains are covered by our treatment for under a change to polar coordinates, they too satisfy Definition 3.4.

## REFERENCES

[**AS**] S. Axler and A. L. Shields, *Univalent multipliers of the Dirichlet space*, Michigan Math. J. **32** (1985), 65–80.

[**DBT**] E. DiBenedetto and N. S. Trudinger, *Harnack inequalities for quasi-minima of variational integrals*, Australian National Univ. Research report, 1984.

[**GT**] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, New York, 1983.

[**JS**] I. Johnstone and M. Shahshahani, *A Poincaré-type inequality for solutions of linear elliptic equations, with statistical applications*, preprint, 1984.

[**MZ**] N. G. Meyers and W. P. Ziemer, *Integral inequalities of Poincaré and Wirtinger type for BV functions*, Amer. J. Math. **99** (1977), 1345–1360.

[**ME**] Norman G. Meyers, *Integral inequalities of Poincaré and Wirtinger type*, Arch. Rational Mech. Anal. **68** (1978), 113–120.

[**P**] J. Polking, *Approximation in L by solutions of elliptic partial differential equations*, Amer. J. Math. **94** (1972), 1231–1244.

[**S**] J. Serrin, *Local behavior of solutions of quasilinear equations*, Acta Math. **111** (1964), 247–302.

[**T**] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.

[**Z**] W. P. Ziemer, *Mean values of subsolutions of elliptic and parabolic equations*, Trans. Amer. Math. Soc. **279** (1983), 555–568.

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405