A POINCARÉ-TYPE INEQUALITY FOR SOLUTIONS OF
ELLIPITIC DIFFERENTIAL EQUATIONS

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ABSTRACT. A sharpened version of the Poincaré inequality is shown to hold
for solutions of a large class of second order elliptic equations.

1. Introduction. If \( \Omega \) is a sufficiently regular domain in \( \mathbb{R}^n \) and \( u \) a smooth
function defined on \( \Omega \), then a well-known form of the Poincaré inequality states
that there is a constant \( C \) such that
\[
(1) \quad \|u - \bar{u}(\Omega)\|_p \leq C\|\nabla u\|_p
\]
where \( 1 \leq p < \infty \) and \( \bar{u}(\Omega) \) denotes the integral average of \( u \) over \( \Omega \). If \( u \) has com-
pact support in \( \Omega \) or vanishes on a set of positive measure, then (1) holds with \( \bar{u}(\Omega) \)
replaced by 0. Inequalities of this type have many applications in analysis, espe-
cially in differential equations, and they admit many variations and generalizations,
cf. [ME, MZ, P, Z].

An interesting result similar to (1) was established in [AS] for holomorphic
(harmonic) functions \( u \) defined on a star-shaped domain in \( \mathbb{C} \) where \( \bar{u}(\Omega) \) is replaced
by the value of \( u \) at any distinguished point in \( \Omega \). In [JS] a similar result was shown
to hold for solutions of second order elliptic equations with smooth coefficients
defined on an open ball. The purpose of this note is to give a rather comprehensive
treatment of this type inequality. Our results are valid for solutions to a large class
of nonlinear equations, including those with measurable coefficients, defined on a
bounded Lipschitz domain \( \Omega \). In the case of linear elliptic equations in divergence
form whose principal coefficients are Hölder continuous, by suitably modifying a
method in [AS], it is shown that \( \Omega \) can be taken as any bounded domain whose
boundary can be expressed locally as the graph of a continuous function.

2. Equations with measurable coefficients. In this section we consider a
bounded, connected Lipschitz domain \( \Omega \) in \( \mathbb{R}^n \). Let \( u \) be an element of the Sobolev
space \( W^{1,p}(\Omega) \) (\( 1 < p < n \)) that is a weak solution of the equation
\[
(2) \quad \text{div} \, A(x \cdot u, \nabla u) = B(x, u, \nabla u)
\]
where \( A \) and \( B \) are Borel measurable vector-valued and scalar-valued functions,
respectively, defined on \( \Omega \times \mathbb{R}^1 \times \mathbb{R}^n \). The function \( u \) is said to be a weak solution
of (2) if
\[
\int A \cdot \nabla \varphi + B \varphi = 0
\]
whenever $\varphi \in W^{1,p}_0(\Omega)$. The functions $A$ and $B$ are required to satisfy the following structural inequalities:

\begin{align*}
|A(x, z, w)| &\leq a_0 |w|^{p-1} + a_1 |z|^{p-1} + a_2, \\
|B(x, z, w)| &\leq b_1 |w|^{p-1} + b_2 |z|^{p-1} + b_3, \\
A(x, z, w) \cdot w &\geq |w|^p - c_1 |z|^p - c_2.
\end{align*}

(3)

Here, $a_0$ is a positive constant and the remaining coefficients are all elements of suitable $L^p(\Omega)$ spaces. The reader may consult [S and T] for a general development of the subject. We will only need that weak solutions $u$ of (2) are locally Hölder continuous ($C^{0,\alpha}$), a fact that follows from the Harnack inequality. The constant $\alpha$ depends only on the structure (3). In particular, $u$ satisfies the following weak Harnack inequality: Let $B(3r)$ denote the ball of radius $3r$ and assume $B(3r) \subset \Omega$. Then

\begin{equation}
\sup_{B(r)} |u| \leq C \left( \int_{B(2r)} |u|^p \right)^{1/p} + K
\end{equation}

where $K$ depends only on $r$ and the structural inequalities (3). Here, $f$ denotes the integral average. It has been shown in [DBT] that (4) remains valid if $p$ is replaced by any $q > 0$. We denote the $L^p$ norm of $u$ on $\Omega$ by $\|u\|_{p;\Omega}$.

2.1 THEOREM. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected Lipschitz domain and let $u \in W^{1,p}(\Omega)$ be a weak solution of (2). For each $x_0 \in \Omega$ there is a constant $C$ that depends only on the structure (3), $\|u\|_{p;\Omega}$, and $\Omega$ such that

\begin{equation}
\|u - u(x_0)\|_{p;\Omega} \leq C \|\nabla u\|_{p;\Omega}.
\end{equation}

PROOF. Assume to the contrary that for every positive integer $j$ there is a weak solution $u_j$ such that

\begin{equation}
\|u_j - u_j(x_0)\|_{p;\Omega} > j \|\nabla u_j\|_{p;\Omega}.
\end{equation}

Although $u_j - u_j(x_0)$ is not necessarily a solution of (2), it is a solution of an associated equation with structure similar to that of (3). For example, if a solution $u$ is replaced by $\breve{u} = u - c$, then $\breve{u}$ satisfies the equation

\begin{equation}
\text{div} \ A(x, \breve{u}, \nabla \breve{u}) = B(x, \breve{u}, \nabla \breve{u})
\end{equation}

with $\overline{A}(x, \breve{u}, \nabla \breve{u}) = A(x, \breve{u} + c, \nabla \breve{u} + c)$ and a similar formula for $\overline{B}$. Clearly the structure for (7) is the same as (3) except for the introduction of $c$ in the coefficients. In our situation, $c = u_j(x_0)$ and $c$ is dominated by an expression involving $\|u_j\|_{p;\Omega}$ by (4). Therefore, by replacing $u_j - u_j(x_0)$ by $u_j$, we may assume that $u_j(x_0) = 0$. Similarly, we may replace $u_j$ by $u_j/\|u_j\|_{p;\Omega}$ and thus assume that $\|u_j\|_{p;\Omega} = 1$. Now we have

\begin{equation}
\|u_j\|_{p;\Omega} > j \|\nabla u_j\|_{p;\Omega}
\end{equation}

with $\|u_j\|_{p;\Omega} = 1$. Therefore, since the Sobolev norms of the $u_j$ are bounded, there exist $u \in W^{1,p}(\Omega)$ and a subsequence of $u_j$ (which we still denote by $u_j$) such that $u_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$. Because $\Omega$ is assumed to be Lipschitz, it follows from an extension of the Kondrachov compactness theorem, cf. [GT, §7.12], that

\begin{equation}
\|u_j - u\|_{p;\Omega} \rightarrow 0.
\end{equation}
It follows from (8) that \( \| \nabla u_j \|_{p;\Omega} \to 0 \) and therefore that \( \| u_j \|_{p;\Omega} \to 1 \). Consequently, \( \| u \|_{p;\Omega} = 1 \) and \( \| \nabla u \|_{p;\Omega} = 0 \). Recall that \( u_j \) is Hölder continuous on each compact subset of \( \Omega \), and by the weak Harnack inequality (4) that the sequence \( \{ u_j \} \) is uniformly bounded. Therefore, by Ascoli’s theorem, a subsequence converges uniformly to \( u \) on each compact subset of \( \Omega \). Thus, \( u \) is continuous on \( \Omega \) and \( u(x_0) = 0 \). Hence \( u \equiv 0 \) on \( \Omega \) because \( \| \nabla u \|_{p;\Omega} = 0 \). This contradicts the fact that \( \| u \|_{p;\Omega} = 1 \).

2.2 REMARK. The dependence of the constant \( C \) on \( \| u \|_{p;\Omega} \) can be avoided if \( A \) and \( B \) are both homogeneous of the same degree in \( z \) and \( w \). In this case \( u_j/\| \nabla u_j \|_{p;\Omega} \) is again a solution and therefore, in the proof above, we may assume that \( \| \nabla u_j \|_{p;\Omega} = 1 \). Since \( \Omega \) is a bounded Lipschitz domain, each \( u_j \) can be extended to all of \( \mathbb{R}^n \) with an equivalent Sobolev norm. Multiplying each \( u_k \) by a smooth function with compact support that is identically 1 on the closure of \( \Omega \) and applying the classical Poincaré inequality implies that the norms \( \| u_j \|_{2;\Omega} \) remain uniformly bounded. Hence, the above proof still applies. For example, the equation

\[
D_i(\| \nabla u \|^{p-2} a^{ij}(x) D_j u) = \| \nabla u \|^{p-1}
\]

is acceptable where \( a^{ij} \in L^\infty(\Omega) \) and \( [a^{ij}] \) is positive definite. In the next section we will see that this dependence can also be avoided if the equation is linear with Hölder continuous coefficients.

3. Linear equations. We will now consider linear equations of the form

\[
Lu = D_i(a^{ij}(x) D_j u + b^i(x) u) + c^i(x) D_i u + d(x) u
\]

whose coefficients are initially assumed to be bounded measurable functions on a bounded, connected domain \( \Omega \subset \mathbb{R}^n \). In addition, it will be assumed that \( L \) is strictly elliptic in \( \Omega \); that is, there exists a positive number \( \lambda \) such that

\[
a^{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^n.
\]

These equations are, of course, a subclass of those considered in §2.

3.1 DEFINITION. For fixed \( x_0 \in \Omega \), let

\[
D_2(\Omega) = W^{1,2}_{\text{loc}}(\Omega) \cap \{ u: Lu = 0, u(x_0) = 0, \| \nabla u \|_{2;\Omega} < \infty \}.
\]

3.2 LEMMA. \( D_2(\Omega) \) is a Banach space.

Proof. Let \( \{ u_i \} \) be a sequence in \( D_2(\Omega) \) such that \( \| \nabla u_i - \nabla u_j \|_{2;\Omega} \to 0 \). Let \( \Omega' \subset \subset \Omega \) be a domain with \( x_0 \in \Omega' \) and let \( \eta \) be a smooth function that is identically 1 on the closure of \( \Omega' \) with \( \text{spt} \eta \subset \Omega \). From the classical Poincaré inequality, it follows that the norms \( \| \eta u_i \|_{2;\Omega} \) are uniformly bounded. Hence, from the weak Harnack inequality (4) we have that the solutions \( u_i \) are uniformly bounded on \( \Omega' \). Because they are Hölder continuous, Ascoli’s theorem implies there is \( u \in W^{1,p}(\Omega') \) such that \( u_i \to u \) uniformly. Thus, \( u(x_0) = 0 \). Elementary considerations show that \( u \in W^{1,p}_{\text{loc}}(\Omega) \) and \( Lu = 0 \).

A Poincaré-type inequality for linear equations with measurable coefficients is a consequence of the following result:

3.3 THEOREM. If \( D_2(\Omega) \subset L^2(\Omega) \), then the inclusion map is continuous.

Proof. We will employ the closed graph theorem to establish the result. To that end, let \( u_i \to u \) in \( D_2(\Omega) \) and let \( u_i \to v \) in \( L^2(\Omega) \). We will show that \( u = v \).
Note that \( D_j u_i \to D_j v \) in \( \mathcal{D}' \) (in the sense of distributions) for any partial derivative \( D_j \) because \( u_i \to v \) in \( \mathcal{D}' \). However, \( D_j u_i \to D_j u \) and, therefore, \( \nabla u = \nabla v \). This implies that \( \nabla v \in L^2(\Omega) \) and \( L u = 0 \). Arguing as in 3.2, we have that \( \{u_i\} \) is uniformly bounded on compact subsets of \( \Omega \) and locally Hölder continuous. An application of Ascoli’s theorem implies that \( u_i \to v \) uniformly on compact subsets and hence, that \( v(x_0) = 0 \). Because \( u(x_0) = 0 \) and \( \nabla u = \nabla v \), it follows that \( u = v \).

The closed graph theorem now applies and the inclusion map is continuous. □

In view of Theorem 3, it is important to determine those domains \( \Omega \) such that \( D_2(\Omega) \subset L^2(\Omega) \). For this purpose we make the following definition.

3.4 DEFINITION. If \( \Omega \) is a bounded domain, we will say that \( \partial \Omega \) is locally a graph if, for each \( x_0 \in \partial \Omega \), there is a neighborhood about \( x_0 \) which is the image, under a rigid motion, of a domain of the form

\[
\{(x',y): x' \in B^{n-1}(r), 0 < y < f(x')\}
\]

where \( B^{n-1}(r) \) is an open ball of radius \( r \) in \( \mathbb{R}^{n-1} \) and \( f: B^{n-1}(r) \to \mathbb{R}^1 \) is continuous and bounded away from 0.

Let us now assume that the coefficients of \( L \) satisfy the following: \( a^{ij}, b^i \in C^{0,\alpha}(\Omega), c^i, d \in L^\infty(\Omega) \).

3.5 THEOREM. If \( \Omega \) is a domain whose boundary \( \partial \Omega \) is locally a graph and \( L \) is as above, then \( D_2(\Omega) \subset L^2(\Omega) \).

PROOF. Because of the assumption on the coefficients of \( L \), it follows that if \( u \in D^2(\Omega) \), then \( u \in C^{1,\alpha}_{\text{loc}}(\Omega) \), cf. [GT, §8.36]. Let \( x_0 \in \partial \Omega \) and consider a neighborhood \( U \) of \( x_0 \) as described in Definition 3.4. For simplicity, assume that \( U \) is of the form (12). For each \( x \in U \), let \( x' \in B^{n-1}(r) \) be such that \( (x', y) = x \). Then

\[
\int_{x'}^x \nabla u(w) \cdot \frac{x - x'}{|x - x'|} dw \\
\leq |x - x'| \int_0^1 |\nabla u(tx + (1 - t)x')| dt.
\]

Hence

\[
|u(x) - u(x')|^2 \leq C(\Omega) \int_0^1 |\nabla u(tx + (1 - t)x')|^2 dt.
\]

Note that

\[
\int_U |u(x) - u(x')|^2 dx \leq C(\Omega) \int_0^1 t^{-1} \int_{tU} |\nabla u|^2 dx dt.
\]

Clearly, closure \( sU \subset U \) for some \( 0 < s < 1 \) and therefore \( |\nabla u|^2 < k \) on \( sU \). Hence, we have

\[
\int_0^s t^{-1} \int_{tU} |\nabla u|^2 dx dt < k|U|
\]

where \( |U| \) denotes the Lebesgue measure of \( U \). Also,

\[
\int_s^1 t^{-1} \int_{tU} |\nabla u|^2 dx dt < \int_s^1 t^{-1} \int_U |\nabla u|^2 dx dt < \infty.
\]

Thus,

\[
\int_U |u(x) - u(x')|^2 dx < \infty.
\]
Since \( u \in C^{1,\alpha}(\Omega) \), \( u \) is bounded on \( B^{n-1}(r) \). This implies that \( \int_U |u(x)|^2 \, dx < \infty \), and because \( \partial \Omega \) can be covered by a finite number of such neighborhoods \( U \), it follows that \( \|u\|_{2,\Omega} < \infty \). \( \Box \)

3.6 REMARK. Of course, Lipschitz domains satisfy Definition 3.4. Likewise, star-shaped domains are covered by our treatment for under a change to polar coordinates, they too satisfy Definition 3.4.

REFERENCES


