PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 2, June 1986

# MOST RIESZ PRODUCT MEASURES ARE $L^p$-IMPROVING

DAVID L. RITTER

ABSTRACT. A Borel measure $\mu$ on a compact abelian group $G$ is *$L^p$-improving* if, given $p > 1$, there is a $q = q(p, \mu) > p$ and a $K = K(p, q, \mu) > 0$ such that $\|\mu * f\|_q \leq K\|f\|_p$ for each $f$ in $L^p(G)$. Here the $L^p$-improving Riesz product measures on infinite compact abelian groups are characterized by means of their Fourier transforms.

**Introduction.** Riesz products have provided harmonic analysts with numerous examples of various phenomena. Here we are interested in characterizing when convolution against Riesz product measures manifests a particular kind of smoothing that we now define.

Let $G$ be an infinite compact abelian group with discrete dual $\Gamma$, and suppose the Haar measures on $G$ and $\Gamma$ are normalized in the usual way so that the Plancherel theorem is valid for the pair. We say a Borel measure $\mu$ defined on $G$ is *$L^p$-improving* if, given $p > 1$, there is a $q = q(p, \mu) > p$ and a $K = K(p, q, \mu) > 0$ such that $\mu$ satisfies

$$\|\mu * f\|_q \leq K\|f\|_p \tag{1}$$

for each $f \in L^p(G)$.

In [**2**], Bonami showed that particular Riesz product measures defined on the circle, on the countable product of circles, and on the Cantor group are $L^p$-improving. Besides providing quantitative results regarding convolution against these measures, she gave a sufficient condition for certain Riesz products defined on an arbitrary compact abelian group $G$ to be $L^p$-improving. What she did not do, however, was to characterize the $L^p$-improving Riesz product measures although a general definition of Riesz product due to Hewitt and Zuckerman was in the literature at that time [**6**].

Our point of departure in this paper is Bonami's work. Here, occasioned by the existence of the more encompassing question raised by Stein in [**8**], we characterize the $L^p$-improving Riesz product measures. Before we do this, though, we recall the definition of the set $R(G)$ of Riesz product measures defined on $G$ in terms of infinite dissociate subsets of $\Gamma$. In recounting the definition, we follow [**3**] or [**5**, Ch. 7], but we use additive notation for the group operation on $\Gamma$.

A subset $\theta$ of $\Gamma$ is said to be *dissociate* if each $\gamma$ in $\Gamma$ can be written uniquely, except possibly for the order of the summands, in the form

$$\gamma = \sum_{j=1}^{n} \varepsilon_j \gamma_j, \qquad \gamma_j \in \theta, \tag{2}$$

Received by the editors June 11, 1985.
1980 *Mathematics Subject Classification*. Primary 43A22; Secondary 43A05, 43A25, 43A46.

with $\varepsilon_j \in \{-1, 1\}$ if $2\varepsilon_j \neq 0$, and $\varepsilon_j = 1$ if $2\gamma_j = 0$. When $a\colon \theta \to \mathbf{C}$ is a function satisfying

$$|a(\gamma)| \leq 1/2 \quad \text{when } 2\gamma \neq 0, \text{ and} \tag{3}$$

$$-1 < a(\gamma) < 1 \quad \text{when } 2\gamma = 0, \tag{4}$$

we set

$$a(\gamma)^\varepsilon = \begin{cases} \overline{a(\gamma)}, & \varepsilon = -1, \\ 1, & \varepsilon = 0, \\ a(\gamma), & \varepsilon = 1. \end{cases}$$

The Reisz product *based on* $\theta$ *and* $a$ is the probability measure $\mu = \mu(a, \theta)$ with Fourier transform defined by

$$\hat{\mu}(\gamma) = \begin{cases} 1, & \gamma = 0; \\ \prod_{j=1}^n a(\gamma_j)^{\varepsilon_j}, & \text{when } \gamma \text{ is as in (2);} \\ 0, & \text{otherwise.} \end{cases}$$

It is well known that $\mu$ can be obtained as a weak* limit of suitably defined polynomials with the limit taken over increasing finite subsets of $\theta$. $R(G)$ is the set of all Riesz products $\mu(a, \theta)$ as $\theta$ runs over all infinite dissociate subsets of $\Gamma$ and $a$ runs through all complex-valued functions $a$ on $\theta$ satisfying (3) and (4).

Having set the stage, we now give our main results.

THEOREM. *Let $\mu$ be in $R(G)$. Then $\mu$ is $L^p$-improving if, and only if, there is a positive number $\delta < 1$ such that the Fourier transform of $\mu$ satisfies*

$$|\hat{\mu}(\gamma)| \leq \delta \tag{5}$$

*for each nonzero $\gamma$ in $\Gamma$.*

Before we prove the theorem, a comment regarding our title is in order. Most Riesz products are $L^p$-improving in the same sense in which most Riesz products are tame: the exceptional cases occur because of the presence of sufficiently many elements of order two in the dual of $G$. The most amusing aspect of this, however, is the fact that the $L^p$-improving Riesz products in $R(G)$ are precisely the tame ones; compare [**3**] and [**5**, Chapter 7].

PROOF OF THE THEOREM. We begin with a standard reduction. A consequence of Young's inequality for measures and the Riesz-Thorin interpolation theorem is this: a measure $\mu$ is $L^p$-improving for *some* $p > 1$ if, and only if, it is $L^p$-improving for *every* $p > 1$. It follows that we can restrict our attention to $q = 2$ in (1).

Our proof of sufficiency requires two lemmas. The first is an easy consequence of the Plancherel theorem and the proof of the general result of Bonami alluded to in our introduction, Theorem 1 of Chapter III of [**2**]. The second lemma is a simple result inspired by Theorem 1 of [**1**] and its proof.

LEMMA 1. *Let $G$ be an infinite compact abelian group with dual $\Gamma$. If $1 < p < 2$, then there are constants $r(p)$ and $K = K(p)$ with $0 < r(p) < 1/2$ and $K > 0$ such that if $\theta$ is a countable dissociate subset of $\Gamma$ and $\mu = \mu(a, \theta)$ is a Riesz product on $G$ with $|a(\gamma)| < r(p)$ for each $\gamma \in \theta$, then $\mu$ satisfies*

$$\|\mu * f\|_2 \leq K\|f\|_p \tag{6}$$

*for each $f$ in $L^p(G)$.*

LEMMA 2. *Let $G$ be an infinite compact abelian group with dual $\Gamma$, and let $\mu$ be a Borel measure on $G$. Suppose there is an $N \geq 1$, a $p' < 2$, and a $K' > 0$ so that the $N$th convolution power of $\mu$, $\mu^N$, satisfies*

$$\|\mu^N * f\|_2 \leq K'\|f\|_{p'} \tag{7}$$

*for each $f$ in $L^{p'}(G)$. Then there is a $p < 2$ and $K > 0$ so that $\mu$ satisfies* (6) *for each $f$ in $L^p(G)$.*

Before proving Lemma 2, we complete the proof of sufficiency.

PROOF OF SUFFICIENCY. Let $p < 2$, and suppose $\mu = \mu(a, \theta)$ satisfies (5) for some $\delta < 1$. Were $\theta$ countable and $\delta < r(p)$ of Lemma 1, no additional work would be needed. If $\theta$ were uncountable and $\delta < r(p)$, a routine argument using the closed graph theorem and Lemma 1 would suffice to show $\mu$ $L^p$-improving. Consequently, in the statement of Lemma 1, we may drop the restriction that $\theta$ be countable. Now, in any event, if $\mu = \mu(a, \theta)$ satisfies (5) for some $\delta < 1$, then for some large $N$, the $N$th convolution power of $\mu$, $\mu^N = \mu(a^N, \theta)$, satisfies the hypotheses of Lemma 1 with arbitrary $\theta$. Consequently, $\mu^N$ is $L^p$-improving. It follows from Lemma 2 that $\mu$ is also $L^p$-improving for possibly a larger $p < 2$. □

PROOF OF LEMMA 2. We will use Stein's analytic interpolation theorem [**9**, p. 205]. For $\mathrm{Re}(z) \geq 0$, define $T^z$ on the simple integrable functions of $G$ by $(T^z f)\hat{}(\gamma) = \hat{f}(\gamma)|\hat{\mu}(\gamma)|^{Nz}$ for $\gamma \in \Gamma$. It is clear that if $f$ is a simple integrable function, $T^z f$ is in $L^2(G)$ for each $z$ since $\hat{\mu}$ is bounded. Consequently, if $f$ and $g$ are fixed simple functions, it follows from

$$\int_G (T^z f) g \, dx = \sum_{\gamma \in \Gamma} \hat{f}(\gamma)\hat{g}(\gamma)|\hat{\mu}(\gamma)|^{Nz}$$

and the Cauchy-Schwarz inequality that $F(z) = \int_G (T^z f) g \, dx$ is analytic for $\mathrm{Re}(z) > 0$. Now if $\mathrm{Re}(z) = 1$, the Plancherel theorem yields

$$\begin{aligned} \|T^z f\|_2^2 &\leq \sum |\hat{\mu}(\gamma)|^{2N} \cdot |\hat{f}(\gamma)|^2 = \sum |(\mu^N)\hat{}(\gamma)|^2 \, |\hat{f}(\gamma)|^2 \\ &= \|\mu^N * f\|_2^2 \leq [K']^2 \|f\|_{p'}^2. \end{aligned}$$

Thus, if $\mathrm{Re}(z) = 1$, $T^z$ is bounded from $L^{p'}(G)$ to $L^2(G)$ with the constant $K'$ independent of $\mathrm{Im}(z)$. Likewise, when $\mathrm{Re}(z) = 0$, we have $\|T^z f\|_2 \leq \|f\|_2$. Stein's theorem now implies that for $0 < t < 1$ there is a $p(t) < 2$ and an $M(t) > 0$ such that $\|T^t f\|_2 \leq M(t)\|f\|_{p(t)}$ for each $f$ in $L^{p(t)}(G)$. Since the Plancherel theorem implies that $\|\mu * f\|_2 = \|T^t f\|_2$ when $t = 1/N$, the lemma is proved. □

We now turn to the proof of necessity and use three lemmas to achieve it. The first of these is due to G. Brown [**3**, Prop. 4], the second is a simple necessary condition for a measure on the dyadic Cantor group $D^\infty$ to be $L^p$-improving, and the third deals with transferring $L^p$-improvement from one group to another.

LEMMA 3. *Let $G = D^\infty$, let $\theta_0 = \{\gamma_j\}_{j=1}^\infty$ be a vector space basis for the dual of $G$ over $\mathbf{Z}/2\mathbf{Z}$, the integers* mod 2, *and let $\mu = \mu(a, \theta_0)$ be a Riesz product based on $\theta_0$ and a function $a$ with $\sum_{j=1}^\infty (1 - |a(\gamma_j)|)$ finite. Then $\mu$ is a discrete measure.*

LEMMA 4. *Let $\mu$ be a Borel measure on $D^\infty$. If there is a $p < 2$ and a $K > 0$ such that $\mu$ satisfies* (6) *for each $f$ in $L^p(D^\infty)$, then $\mu$ is continuous.*

LEMMA 5. *Let $G$ and $H$ be compact abelian groups with Haar measures $m_G$ and $m_H$ normalized to have total mass 1. Suppose $\pi$ is a continuous homomorphism from $G$ onto $H$. When $\mu$ is a Borel measure on $G$, let $\pi\mu$ be the Borel measure on $H$ defined by*

$$\int_H f\,d\pi\mu = \int_G f\circ\pi\,d\mu, \qquad f\in C(H), \tag{8}$$

*where $C(H)$ denotes the continuous, complex-valued functions on $H$. If there is a $q>p$ and $K>0$ so that $\mu$ satisfies* (1) *for each $f$ in $L^p(G)$, then for $p,q$, and $K$ as in the hypothesis,* (1) *is true with $\mu$ replaced by $\pi\mu$ for each $f$ in $L^p(H)$.*

Before proving Lemma 4 and Lemma 5, we complete the proof of necessity.

PROOF OF NECESSITY. Suppose $\mu=\mu(a,\theta)$ is a Riesz product in $R(G)$ such that for each $\delta<1$ there is a $\gamma\in\Gamma$ with $|\hat\mu(\gamma)|>\delta$. It follows that there is a sequence $\theta_0=\{\gamma_j\}_{j=1}^\infty$ of elements of $\theta$ of order two with the sum $\sum_{j=1}^\infty(1-|a(\gamma_j)|)$ finite. Since $\theta$ is dissociate and each element of $\theta_0$ has order two, $\theta_0$ is a vector space basis over $\mathbf{Z}/2\mathbf{Z}$ for the subgroup $\Gamma_0$ of $\Gamma$ it generates. Let $G_0$ be the annihilator in $G$ of $\Gamma_0$. Then $H=G/G_0$ is topologically isomorphic to $D^\infty$, and if $\pi\colon G\to H$ is the natural map and $\tilde\Gamma_0$ is the set of characters on $H$, then $\tilde\Gamma_0$ and $\Gamma_0$ are topologically isomorphic via the adjoint map $\pi^*\colon\tilde\Gamma_0\to\Gamma_0$ defined by $\pi^*(\tilde\gamma)=\tilde\gamma\circ\pi$. It follows then from equation (8) that $\pi\mu$ is the Riesz product on $H$ based on $\tilde\theta_0$ and $\tilde a$, where $\tilde\theta_0=\{\tilde\gamma_j\colon\pi^*(\tilde\gamma_j)=\gamma_j,\ j\ge1\}$ and $\tilde a$ is the function defined by $\tilde a(\tilde\gamma_j)=a(\gamma_j)$. Since $\tilde\theta_0$ is a basis for $\tilde\Gamma_0$ and the sum $\sum_{j=1}^\infty(1-|\tilde a(\tilde\gamma_j)|)$ is finite, Lemma 3 shows that $\pi\mu$ is discrete. Consequently, from Lemma 4, we see $\pi\mu$ is not $L^p$-improving on $H$. Lemma 5 then implies $\mu$ is not $L^p$-improving on $G$. That concludes the proof of necessity using Lemmas 3, 4, and 5. □

PROOF OF LEMMA 4. Let $\{\gamma_j\}_{j=1}^\infty$ be a basis for the dual of $G,\Gamma$. As is well known [**4**, Ch. 14], for $n\ge1$ each of the polynomials $f_n$ defined by

$$f_n(x)=\prod_{j=1}^n(1+\gamma_j(x))=\sum_{\varepsilon\in\{0,1\}^n}\left[\sum_{j=1}^n\varepsilon_j\gamma_j(x)\right]$$

is $2^n$ times the characteristic function of the subgroup $G_n=\{x\in G\colon\gamma_j(x)=1$ for $j=1,\dots,n\}$ of $G$. Since $G_n$ has total Haar measure $2^{-n}$, if $1<p<2$ and $1/p+1/p'=1$, we have $\|f_n\|_p=2^{n/p'}$. By convolving $f_n$ with $\mu$ and using (6), we obtain

$$\sum_{\varepsilon\in\{0,1\}^n}\left|\hat\mu\left(\sum_{j=1}^n\varepsilon_j\gamma_j\right)\right|^2\le K^2\|f_n\|_p^2\le K^2\cdot2^{2n/p'}.$$

Dividing both sides of this inequality by $2^n$, letting $n\to\infty$, and applying the general form of Wiener's theorem as found in [**7**, p. 118] or [**5**, p. 415], we obtain the proof of the lemma. □

PROOF OF LEMMA 5. Since equation (8) is true with $\pi\mu$ replaced by $m_H$ and $\mu$ replaced by $m_G$, if $f$ is a continuous function defined on $H$, a routine computation reveals that

$$\|\pi\mu*f\|_{L^q(H)}^q=\|\mu*(f\circ\pi)\|_{L^q(G)}^q\le K^q\|f\circ\pi\|_{L^p(G)}^q=K^q\|f\|_{L^p(H)}^q.$$

It follows that (1) is true with $\mu$ replaced by $\pi\mu$ for $f$ in $C(H)$. Since $C(H)$ is dense in $L^p(H)$, the lemma is proved. □

**A concluding remark.** The original definition of Riesz product given by Hewitt and Zuckerman in [**6**] allowed equality to hold in (4). With (4) replaced by

$$(4') \qquad -1 \leq a(\gamma) \leq 1 \quad \text{when } 2\gamma = 0,$$

the statement of our theorem needs only a slight alteration. It should now read

THEOREM. *Let $\mu$ be a Riesz product measure. Then $\mu$ is $L^p$-improving if, and only if, there is a positive number $\delta < 1$ and a finite subset $F$ of $\Gamma$ containing zero such that the Fourier transform of $\mu$ satisfies* (5) *for each $\gamma \in \Gamma \backslash F$.*

The proof of necessity is essentially the same as before, and it is an easy exercise to reduce sufficiency to the previous proof of sufficiency.

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DEPARTMENT OF MATHEMATICAL SCIENCES, FLORIDA INTERNATIONAL UNIVERSITY, MIAMI, FLORIDA 33199