A NEW INEQUALITY FOR COMPLEX-VALUED POLYNOMIAL FUNCTIONS

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ABSTRACT. Let $f_1, f_2, \ldots, f_n : \mathbb{C} \rightarrow \mathbb{C}$ be complex-valued polynomial functions of degrees $d_1, d_2, \ldots, d_n$, respectively, of a complex variable $z$. Then

$$M_{f_1} M_{f_2} \cdots M_{f_n} \geq M_{f_1 f_2 \cdots f_n} \geq k M_{f_1} M_{f_2} \cdots M_{f_n}$$

where

$$k = \left( \sin \frac{\pi}{n 8d_1} \right)^{d_1} \left( \sin \frac{\pi}{n 8d_2} \right)^{d_2} \cdots \left( \sin \frac{\pi}{n 8d_n} \right)^{d_n}.$$ 

Let $\mathbb{C}$ be the field of complex numbers and $f : \mathbb{C} \rightarrow \mathbb{C}$ be a complex-valued polynomial function of a complex variable $z$. Define $M_f = \max_{|z|=1} |f(z)|$.

THEOREM. Let $f_1, f_2, \ldots, f_n : \mathbb{C} \rightarrow \mathbb{C}$ be polynomial functions of degrees $d_1, d_2, \ldots, d_n$, respectively, of a complex variable $z$. Then

$$M_{f_1} M_{f_2} \cdots M_{f_n} \geq M_{f_1 f_2 \cdots f_n} \geq k M_{f_1} M_{f_2} \cdots M_{f_n}$$

where

$$(1) \quad k = \left( \sin \frac{\pi}{n 8d_1} \right)^{d_1} \left( \sin \frac{\pi}{n 8d_2} \right)^{d_2} \cdots \left( \sin \frac{\pi}{n 8d_n} \right)^{d_n}.$$ 

PROOF. Of course the left inequality is obvious.

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an arbitrary polynomial function of degree $d$ with leading coefficient one, among the given functions \{f_i\}. Then we can factor $f(z)$ in the form

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_d),$$

so that

$$M_f = \max_{|z|=1} |f(z)| = \max_{|z|=1} |z - z_1| |z - z_2| \cdots |z - z_d|$$

$$\leq (1 + |z_1|)(1 + |z_2|) \cdots (1 + |z_d|).$$

We will prove that for each $\varepsilon > 0$ and $n > 0$, we have for most $\theta$ that

$$(3) \quad |f(e^{i\theta})| = |e^{i\theta} - z_1| |e^{i\theta} - z_2| \cdots |e^{i\theta} - z_d| \geq M_f \left( \sin \frac{\pi}{n 8d + \varepsilon} \right)^d.$$

However by (2), we see that it is enough to show that

$$(4) \quad \frac{|e^{i\theta} - z_1| |e^{i\theta} - z_2| \cdots |e^{i\theta} - z_d|}{(1 + |z_1|)(1 + |z_2|) \cdots (1 + |z_d|)} \geq \left( \sin \frac{\pi}{n 8d + \varepsilon} \right)^d.$$
for all values of $\theta$ in the complement of the union $U_d$ of intervals on $[-\pi, \pi]$ with total length $l(U_d) \leq (2/n)(8d\pi)/(8d + \varepsilon)$. Let $z$ be an arbitrary element in $\{z_i\}$ and set $z = re^{i\alpha}$, $r > 0$, and $\phi = (2/n)\pi/(8d + \varepsilon)$. Then for $\theta$ satisfying the inequality $2\phi \leq |\theta| \leq \pi - 2\phi$, we have $(\rho = 2r/(r^2 + 1) < 1)$

$$
\frac{|1 + re^{i\theta}|^2}{(1 + r)^2} = \frac{1 + r^2 + 2r \cos \theta}{1 + r^2 + 2r} \geq \frac{1 + r^2 - 2r \cos 2\phi}{1 + r^2 + 2r}
$$

$$
= \frac{1 - (2r/(1 + r^2)) \cos 2\phi}{1 + (2r/(1 + r^2))} = \frac{1 - \rho \cos 2\phi}{1 + \rho} \geq \frac{1 - \cos 2\phi}{2} = \sin^2 \phi.
$$

Thus $|1 + re^{i\theta}|/(1 + r) \geq \sin \phi$ for all values of $\theta$ in the complement of the union $V$ of intervals with total length $l(V) \leq 8\phi$. Therefore, on a similar set we have that

$$
|e^{i\theta} - re^{i\alpha}|/(1 + |re^{i\alpha}|) \geq \sin \phi
$$

and so (4) follows. Applying the estimate (2) for each polynomial function $f_i$ of degree $d_i$, we obtain $n$ inequalities of the form (3) (with $f = f_1, f_2, \ldots, f_n$), with each holding for all values of $\theta$ in the complement of the union $U_d$ of intervals with total length

$$
l(U_d) \leq (2/n)(8d\pi)/(8d + \varepsilon),
$$

respectively, for $i = 1, 2, \ldots, n$.

We add this for all $n$, and deduce that all $n$ inequalities (3) hold for some

$$
\beta \in [-\pi, \pi] - \bigcup_{i=1}^{n} U_{d_i}.
$$

Therefore

$$
M_{f_1, f_2 \cdots f_n} \geq |f_1(e^{i\beta}) \cdot f_2(e^{i\beta}) \cdots f_n(e^{i\beta})|
$$

$$
= |f_1(e^{i\beta})||f_2(e^{i\beta})| \cdots |f_n(e^{i\beta})| \geq M_{f_1} \cdot M_{f_2} \cdots M_{f_n}
$$

$$
\cdot \left(\sin \frac{2 \pi}{n \ 8d_1 + \varepsilon}\right)^{d_1} \left(\sin \frac{2 \pi}{n \ 8d_2 + \varepsilon}\right)^{d_2} \cdots \left(\sin \frac{2 \pi}{n \ 8d_n + \varepsilon}\right)^{d_n}.
$$

Taking the limit as $\varepsilon \to 0$ (6) implies

$$
M_{f_1, f_2 \cdots f_n} \geq k \cdot M_{f_1} M_{f_2} \cdots M_{f_n}
$$

where $k$ satisfies (1).

**REMARK.** See [1-5] for other results and some research problems concerning inequalities on the derivative of a complex-valued function.

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**REFERENCES**


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