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MULTIPLE IMAGES AND LOCAL TIMES OF MEASURABLE FUNCTIONS[1]

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ABSTRACT. Let $x(t)$, $0 \le t \le 1$, be a real-valued measurable function having a local time $\alpha_{[0,t]}(x)$ which is continuous in t, for almost all x. Then, for every integer $m \ge 2$, and every nonempty open subinterval $J \subset [0,1]$, there exist m disjoint subintervals $I_1, \ldots, I_m \subset J$ such that the intersection of the images of $I_1, \ldots, I_m$ under the mapping $t \to x(t)$ has positive Lebesgue measure. The result applies to a large class of sample functions of stochastic processes, and also to multidimensional t and $x(\cdot)$.

1. Introduction. Let $x(t)$, $t \in J$, be a real-valued measurable function defined on a real interval J having a nonempty interior. For an arbitrary subinterval $I \subset J$, put $x(I) =$ image of I. Following our terminology in [**2**], we say that $x(\cdot)$ has a multiple image of order m over J, where $m \ge 2$ is an integer, if there are m disjoint subintervals in J such that $\bigcap_{i=1}^m x(I_i)$ has positive Lebesgue measure. The concept is directly extendable to vector-valued functions of several variables. In [**2**] we discuss the multiple image property of the sample functions of certain stochastic processes; in particular, they have the property of having a multiple image over every subinterval J of their domain of definition. Such functions are extremely irregular because they repeat substantial proportions of their values within arbitrarily small intervals.

Our main result here is a purely analytic one, and does not depend on a probabilistic structure. We recall the definition of the local time of a real measurable function $x(t)$, $0 \le t \le 1$. For every pair of measurable sets $A \subset R^1$ and $I \subset [0,1]$, define $\nu(A, I) =$ Lebesgue measure of $(s\colon\ x(s) \in A, s \in I)$. If, for fixed I, $\nu(\cdot, I)$ is absolutely continuous as a measure of sets A, then its Radon-Nikodym derivative, which we denote as $\alpha_I(x)$, is called the local time of $x(t)$ relative to I. By definition, it satisfies

$$\nu(A, I) = \int_A \alpha_I(x)\, dx, \quad \text{for all } A, I. \tag{1.1}$$

An elementary argument [**2**] shows that, for arbitrary disjoint intervals $I_1, \ldots, I_m$, the images $x(I_1), \ldots, x(I_m)$ have an intersection of positive measure if the local time satisfies

$$\alpha_{I_1}(x) \cdots \alpha_{I_m}(x) > 0 \tag{1.2}$$

on a set of positive measure.

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As in [**1**], we say that the local time is temporally continuous if the function $\alpha_{[0,t]}(x)$ is continuous in t for almost all x. Our main result is

THEOREM 1. *Let $x(t)$, $0 \le t \le 1$, have a local time which is temporally continuous. Then, for every nonempty open interval $J \subset [0,1]$, $x(t)$ has a multiple image of order m over J for every $m \ge 2$.*

This theorem represents another manifestation of the relation between the smoothness of the local time and the irregularity of the underlying function $x(\cdot)$, which has been the theme of the author's research in this area. A recent result of this type is that, under the hypothesis of temporal continuity of the local time, almost no point t is a point of increase of the function $x(t)$ [**1**].

This result is particularly applicable to the sample functions of certain classes of stochastic processes, where the relevant properties of the local time can be obtained from the finite-dimensional distributions of the process. Theorem 1 represents a generalization of my earlier result [**2**]. There m was restricted to be an integer power of 2, and certain continuity and integrability conditions were assumed to hold for the m-dimensional joint density of the process for the given index m of the multiple image. The condition that m be a power of 2 was relaxed by Shieh [**4**], but the index was still tied to the condition on the joint density. The present result, which is valid for all $m \ge 2$, holds for a large class of processes whose bivariate distributions satisfy a simple condition. For example, if $p(x,y;s,t)$ is the bivariate density of the process at the time points s, t with $s \ne t$, then the local time exists and is temporally continuous if the function

$$\int_0^1 \int_0^1 p(x,y;s,t)\,ds\,dt$$

is continuous in (x,y), at least for $x = y$. See the survey of Geman and Horowitz [**3**] for other conditions of this type.

2. Proof of the theorem. Define

$$\alpha_{n,j}(x) = \alpha_{[(j-1)2^{-n},j2^{-n}]}(x), \tag{2.1}$$

for $j = 1,\dots,2^n$, and $n \ge 1$. Then, as in [**2**], we have

$$\alpha_{[0,1]}(x) = \sum_{j=1}^{2^n} \alpha_{n,j}(x), \quad \text{a.e. } x. \tag{2.2}$$

Thus, for arbitrary $m \ge 2$,

$$\alpha^m_{[0,1]}(x) = \sum_{j_1,\dots,j_m=1}^{2^n} \prod_{i=1}^{m} \alpha_{n,j_i}(x). \tag{2.3}$$

Consider the subset of the terms of the sum in (2.3) for which the first two indices are equal, that is, $j_1, = j_2$. It follows from (2.1) that the sum of the terms in this subset is equal to

$$\alpha^{m-2}_{[0,1]}(x) \sum_{j=1}^{2^n} \alpha^2_{n,j}(x). \tag{2.4}$$

Under the hypothesis of temporal continuity, the sum of squares, $\sum_{j=1}^{2^n} \alpha_{n,j}^2(x)$, converges to 0, a.e. x, because $\alpha_{[0,1]}(x)$ is bounded, monotonic and continuous in t. Therefore, the expression (2.4) converges to 0, a.e. x. Hence the terms of the sum (2.3) for which $j_1 = j_2$ may be omitted in the computation of the left-hand member of (2.3) as the limit, for $n \to \infty$, of the right-hand member. By the same logic, we may omit all terms in the sum in (2.3) for which any two indices are equal. Therefore, the sum (2.3) has the same limit as when the summation is restricted to terms for which the indices are all distinct:

$$\alpha_{[0,1]}^m(x) = \lim_{n\to\infty} \sum_{j_1,\dots,j_m=1, j_1,\dots,j_m \text{ distinct}}^{2^n} \prod_{i=1}^{m} \alpha_{n,j_i}(x). \tag{2.5}$$

Now the left-hand member of (2.5) must be positive on a set of positive Lebesgue measure; indeed, this follows from

$$\int_{R^1} \alpha_{[0,1]}(x)\,dx = 1,$$

which is a consequence of (2.1). Therefore, the limit of the right-hand member is also positive on a set of positive Lebesgue measure. Therefore, for each n sufficiently large, at least one of the terms $\prod_{i=1}^m \alpha_{n,j_i}(x)$ is positive on a set of positive measure, and so, by (1.2), there is a multiple image of order m over $[0,1]$.

This establishes the theorem for the interval $J = [0,1]$. The reasoning is valid for any $J \subset [0,1]$, and so the proof is complete.

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