EXACT SEQUENCES OF SPECTRA AND DUALITY

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Abstract. We note that duality arguments show that, under certain circumstances, an exact sequence of spectra remains exact after smashing with a finite complex. Among various applications, we show that the validity for $S^0$ of G. W. Whitehead's symmetric products conjecture implies that the conjecture is "almost" true for all finite complexes.

1. Introduction. In [6], we studied sequences of spectra \( \cdots \to E_2 \to E_1 \to E_0 \) having the property that application of the 0th space functor \( \Omega^0 \) yields a sequence of spaces of the form \( \cdots \to X_2 \times X_1 \to X_1 \times X_0 \to X_0 \), with all maps being the obvious composites of projections and inclusions. Such an exact sequence induces a long exact sequence on homotopy groups in nonnegative degrees.

In this note, we observe that a simple duality argument shows that exact sequences often remain exact after being smashed with a finite complex. Thus there is more to be gleaned from knowing that a sequence is exact than might at first be apparent.

As applications of this principle, we offer the following three examples (which will be elaborated upon in §3).

In [4], D. Kahn and S. Priddy showed that, localized at a prime \( p \), there is a stable map \( \Sigma^\infty \Sigma_p \to \Sigma^\infty S^0 \) such that \( \Omega^\infty \lambda : Q\Sigma_p \to Q_0 S^0 \) is the projection onto a direct factor. Here \( QX = \Omega^\infty \Sigma^\infty X \), and \( Q_0 S^0 \) is the component of the basepoint. Thus \( \pi_q^*(Q\Sigma_p) \to \pi_q^*(S^0) \) is onto for \( q > 0 \). We use this to show that if \( Y \) is a finite complex and \( n = 2 \dim Y + 1 \), then \( \Omega^n Q(\Sigma P_\eta Y \land Y) \to \Omega^n QY \) is also the projection onto a direct factor. Thus \( \pi_q^*(\Sigma P_\eta Y \land Y) \to \pi_q^*(Y) \) is onto for \( q > 2 \dim Y \) (this has been independently observed by J. Jones [3]).

In [5, 7], the Kahn-Priddy epimorphism was extended to a long exact sequence to conclude that, localized at \( p \),

\[
(1.1) \quad \ker \left\{ \pi_q(SP^n Y) \to \pi_q(SP^{n+1} Y) \right\} = \ker \left\{ \pi_q(SP^n Y) \to \pi_q(SP^p Y) \right\}
\]

is valid when \( Y = S^0 \). Here \( SP^n E \) denotes the \( m \)th symmetric product of a spectrum \( E \). This was originally conjectured to be true for all \( Y \) by G. W. Whitehead [9], but a counterexample was found by P. Welcher [11] when \( p = 2, k = 0, q = 5 \) and \( Y = S^0 \cup e^4 \). In spite of this, we use the result for \( S^0 \) to show that if \( Y \) is a finite complex, then (1.1) \( \textit{is} \) true for all but a finite number of exceptional pairs \( (k, q) \).
As a final example, recall that J. Becker showed that there is an infinite loop map \( QBO(2) \rightarrow BO \) which is the projection onto a direct factor \([2]\). Using this, we show that there is a similar projection \( \Omega^6Q(BO(2) \wedge CP^2) \rightarrow BU \times \mathbb{Z} \).

2. Exact sequences and duality. We recall some definitions from \([6]\). A spectrum \( P \) is spacelike if it is a wedge summand of a suspension spectrum. A fibration sequence \( A \rightarrow B \rightarrow C \) is short exact if \( \Omega^\infty f : \Omega^\infty B \rightarrow \Omega^\infty C \) is the projection onto a direct factor, so that \( \Omega^\infty B = \Omega^\infty A \times \Omega^\infty C \). The map \( f \) is said to be onto.

A sequence of spectra \( \cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow E_0 \) is exact if it is obtained from a diagram of the form

\[
\begin{array}{cccc}
\cdots & E_2 & \cdots & E_1 & \cdots & E_0 \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow \\
X_2 & X_1 & X_0 & \\
\end{array}
\]

where each \( E_k \rightarrow X_k \rightarrow E_k \) is short exact.

For the remainder of this paper, we will assume that all spectra are C. W. spectra of finite type (possibly localized at \( p \)). Thus any spectrum \( E \) considered will be the direct limit of a sequence of finite spectra.

The following is the key observation of this paper.

**Theorem 2.1.** Suppose that \( \cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow E_0 \) is an exact sequence of spectra and that \( \pi_q(X_k) \) is finite for \( q \geq 0 \) and \( k \geq 1 \). If \( Y \) is a finite complex such that the \( n \)-dual of \( Y \) is spacelike, then there is an exact sequence

\[
\cdots \rightarrow \Sigma^{-n}X_2 \wedge Y \rightarrow \Sigma^{-n}X_1 \wedge Y \rightarrow \Sigma^{-n}X_0 \wedge Y \rightarrow \Sigma^{-n}E_0 \wedge Y.
\]

Note that, if \( Y \) is \((-1\)-connected, any \( n \geq 2 \dim Y \) satisfies the conditions.

**Proof.** To begin, it suffices to prove the theorem with the long exact sequence replaced by a short exact sequence \( A \rightarrow B \rightarrow C \) such that \( \pi_q(A) \) is finite for \( q \geq 0 \).

Standard arguments (see \([6]\)) using the adjoint functors \((\Sigma^\infty, \Omega^\infty)\) show that \( \Sigma^{-n}A \wedge Y \rightarrow \Sigma^{-n}B \wedge Y \rightarrow \Sigma^{-n}C \wedge Y \) will be exact if and only if

\[
0 \rightarrow \lim[H, \Sigma^{-n}A \wedge Y] \rightarrow \lim[H, \Sigma^{-n}B \wedge Y] \rightarrow \lim[H, \Sigma^{-n}C \wedge Y] \rightarrow 0
\]

is exact for all spacelike spectra \( P \).

Recall the duality isomorphism \([X \wedge DY, Z] = [X, Z \wedge Y] \), valid if both \( X \) and \( Y \) are finite \([1, p. 195]\). Thus the exactness of \( A \rightarrow B \rightarrow C \) implies that (2.2) is exact for all finite spacelike \( P \), since \( \Sigma^nP \wedge DY \) will be spacelike.

For the general case, \( P = \lim P_i \), where the \( P_i \) are finite spacelike spectra. There is a diagram

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
\lim[H, \Sigma P_i, \Sigma^{-n}A \wedge Y] & \lim[H, \Sigma P_i, \Sigma^{-n}B \wedge Y] & \lim[H, \Sigma P_i, \Sigma^{-n}C \wedge Y] & 0 \\
\downarrow & \downarrow & \downarrow \\
[P, \Sigma^{-n}A \wedge Y] & [P, \Sigma^{-n}B \wedge Y] & [P, \Sigma^{-n}C \wedge Y] \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

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where the vertical sequences are Milnor exact sequences, and the top and bottom ones are part of standard six-term $\lim^1$ exact sequences. Under our assumption about $\pi_*(A)$, the sets $[\Sigma P_i, \Sigma^{-n}A \wedge Y]$ and $[P_i, \Sigma^{-n}A \wedge Y]$ are finite. The corresponding $\lim^1$ terms thus vanish, and the exactness of the middle horizontal sequence follows.

3. Examples. Our first applications of Theorem 2.1 are to the exact sequence

$$\cdots \to \Sigma L(2) \to \Sigma L(1) \to \Sigma L(0) \to \Sigma HZ$$

constructed in our solution to Whitehead's conjecture for $S^0$. Here all spectra are to be localized at $p$, $L(k) = \Sigma^{-k}SP^pS^0/SP^pS^0$, and $HZ$ is the integral Eilenberg-Mac Lane spectrum (which can be identified with $SP^\infty S^0$ by the Dold-Thom Theorem). $L(0) = \Sigma^\infty S^0$ and $L(1) = \Sigma^\infty B\Sigma_p$, so exactness at $\Sigma L(0)$ is the Kahn-Priddy Theorem. (The extra suspension is a bit of a bonus.)

For $k \geq 1$, $L(k)$ has finite homotopy groups, since, for example, it is known to be a stable wedge summand of $B(\mathbb{Z}/p)^k$ [12]. Thus Theorem 2.1 can be directly applied to conclude that, for every finite complex $Y$, there is an exact sequence

$$\cdots \to \Sigma^{1-n}L(2) \wedge Y \to \Sigma^{1-n}L(1) \wedge Y \to \Sigma^{1-n}Y \to \Sigma^{1-n}HZ \wedge Y$$

with $n$ chosen so that the $n$-dual of $Y$ is spacelike.

Consideration of the beginning of this sequence yields the next theorem.

**Theorem 3.2.** Let $Y$ be a finite complex with $n$ chosen as above. Let

$$M = \max \{ n - 1, \dim Y + 1 \}.$$

(a) $\pi_q^i(B\Sigma_p \wedge Y) \to \pi_q^i(Y)$ is onto for $q \geq M$.
(b) Let $\alpha: \Sigma^d Y \to Y$ be any stable self-map. If $dN > M$ then the $N$th iterate of $\alpha$ lifts:

$$\begin{array}{c}
B\Sigma_p \wedge Y \\
\downarrow \\
\Sigma^d Y \\
\alpha^N \\
\downarrow \\
Y
\end{array}$$

(c) $\Omega^MQ(B\Sigma_p \wedge Y) \to \Omega^MQY$ is the projection onto a direct factor.

All of these follow immediately from the exactness of (3.1) at $\Sigma^{1-n}Y$ together with the observation that $\Omega^MQ^{\infty}(HZ \wedge Y) = \star$ since $M > \dim Y$.

Exactness of the whole sequence (3.1) can be used to attack Whitehead's conjectures.

**Theorem 3.3.** With $Y$ and $n$ as above, (1.1) is true unless $k + n - 1 > q \geq 2p^{k+1} - 2$.

**Proof.** The key is the observation that, for a spectrum $E$, $SP^mE = SP^mS^0 \wedge E$ [11]. Straightforward diagram chasing applied to (3.1) then gives the stated upper bound on $q$. The lower bound is trivial: connectivity arguments show that $\pi_q(SP^k Y) \to \pi_q(SP^\infty Y)$ is an isomorphism if $q < 2p^{k+1} - 2$. 

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Examples 3.4.

<table>
<thead>
<tr>
<th>$Y$</th>
<th>$n$</th>
<th>$(k, q)$ satisfying the inequalities of Theorem 3.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^0 \cup \eta e^1$</td>
<td>2</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$S^0 \cup \eta e^2$</td>
<td>4</td>
<td>(0, 2)</td>
</tr>
<tr>
<td>$S^0 \cup \eta^2 e^3$</td>
<td>5</td>
<td>(0, 2), (0, 3)</td>
</tr>
<tr>
<td>$S^0 \cup p e^4$</td>
<td>8</td>
<td>$(0, q), \quad 2 \leq q &lt; 7, \quad (1, q), \quad 6 \leq q &lt; 16,$</td>
</tr>
<tr>
<td>$S^0 \cup e^8$</td>
<td>16</td>
<td>$(2, q), \quad 14 \leq q &lt; 17$</td>
</tr>
</tbody>
</table>

When the attaching map is $\eta$ or $\eta^2$, easy diagram chasing eliminates the exceptional pairs $(k, q)$. (This includes Welcher’s mod 2 examples [11, Corollary 6.5].)

Example 3.5. Let $p$ be odd and let $\alpha \in \pi_{2p-3}(S^0)$ be a generator. Then (1.1) is true for $Y = S^0 \cup \alpha e^{2p-2}$. In this case, the inequalities leave open only the cases $(k, q) = (0, 2p - 2)$ or $(0, 2p - 1)$. Again it is easy to eliminate these cases.

For our next application of Theorem 2.1, we interpret Becker’s result about $BO$ as the construction of an onto map $f: \Sigma^\infty BO(2) \to ko\langle 0 \rangle$, where $ko\langle 0 \rangle$ is the 0-connected cover of the periodic real $K$-theory spectrum $KO$. Since the fiber of $f$ has finite homotopy groups, Theorem 2.1 applies. Thus letting $Y = S^0 \cup e^2$, there is an onto map

\[(3.6) \quad \Sigma^{-4} \Sigma^\infty BO(2) \land Y \to \Sigma^{-4} ko\langle 0 \rangle \land Y.\]

Now note that $Y = \Sigma^{-2} CP^2$, $\pi_q(\Sigma ko\langle 0 \rangle \land Y) = \pi_q(KO \land Y)$ for $q > 2$, and $KO \land Y = KU = \Sigma^{-4} KU$ [1, p. 206]. These facts, together with (3.6), imply that there is an onto map $\Sigma^{-6} \Sigma^\infty BO(2) \land CP^2 \to KU$.

The next theorem follows.

**Theorem 3.7.** There is an infinite loop map $\Omega^6 Q(BO(2) \land CP^2) \to BU \times \mathbb{Z}$ which is a projection onto a direct summand.

**Remark 3.8.** In using Theorem 2.1, it occasionally happens that $n$ can be chosen to be quite small. As an amusing example, let $Y = B(k)$, the $k$th mod 2 Brown-Gitler spectrum with $H^*(B(k)) = A/A\{\chi(Sq^i) | i > k \}$. It was conjectured by H. Miller [10] and proved by J. Lannes [8] that the $2k$th dual of $B(k)$ is spacelike. Thus we conclude that a short exact sequence of spectra $A \to B \to C$ induces a short exact sequence

\[0 \to B(k)_q(A) \to B(k)_q(B) \to B(k)_q(C) \to 0\]

for $q \geq 2k$. Note that this range is essentially complementary to the range of $q$ for which properties of $B(k)$ allow one to do computations.

**References**


