EXTENSION OF CONTINUOUS FUNCTIONS INTO UNIFORM SPACES

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ABSTRACT. Let X be a dense subspace of a topological space T, let Y be a uniformizable space, and let f: X → Y a continuous map. In this paper we study the problem of the existence of a continuous extension of f to the space T. Thus we generalize basic results of Taimanov, Engelking and Blefko-Mrówka on extension of continuous functions. As a consequence, if D is a nest generated intersection ring on X, we obtain a necessary and sufficient condition for the existence of a continuous extension to v(X, D), of a continuous function over X, when the range of the map is a uniformizable space, and we apply this to realcompact spaces. Finally, we suppose each point of T\X has a countable neighbourhood base, and we obtain a generalization of a theorem by Blair, herewith giving a solution to a question proposed by Blair.

In the sequel the word space will designate a topological space. By µX we shall denote a uniform space, µ being a collection of covers in the sense of Tukey [13] and Isbell [10]. γµX will mean the completion of the space µX. Let µ be a uniformity on X and let m be an infinite cardinal number, then µ_m will denote the uniformity on X generated by the µ-covers of power < m (we only consider µ and m for which µ_m is definable; see [6, 10 and 14]). The compact uniform space γµ_{ro}X is called the Samuel compactification of µX, denoted by sµX. By τµX we shall denote the set X equipped with the µ-uniform topology. We say that E and D subsets of X are µ-separated when E is far from D in the proximity defined by µ.

By a base on a space X we mean a nest generated intersection ring (or equivalently, a strong delta normal base) in X [1 and 10]. It is known that each base D on X has associated a Wallman compactification W(X, D) and a Wallman real-compactification v(X, D). If µX is a uniform space, we shall denote by Z(µX) the base on τµX formed by uniform zero-sets (see [7 and 8]), v(µX) will denote the space v(X, Z(µX)) and β(µX) the space W(X, Z(µX)).

Let T be a space and suppose X ⊆ T. If m is a cardinal number, we say that X is m-dense in T when for every family {Ui: i ∈ I}, with |I| < m, of nonvoid open sets in T such that \( \bigcap \{U_i: i \in I\} \neq \emptyset \), it follows that \( \bigcap \{U_i \cap X: i \in I\} \neq \emptyset \). If A is a subset of T, we define the m-closure of A in T as the union of all subspaces of T in which A is m-dense.

The following result generalizes the Taimanov theorem [12] of extension of continuous functions into compact spaces.

**Theorem 1.** Let T be a space in which X is m-dense, let µY be a uniform space, and let f: X → τµY a continuous map. Then the following are equivalent:

(a) f has a continuous extension \( f: T \to \tau \gamma \mu_m Y \).

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(b) For every pair $E$ and $D$ of subsets of $Y$ which are $\mu$-separated, we have that $\text{cl}_T f^{-1}(E) \cap \text{cl}_T f^{-1}(D) = \emptyset$, i.e., $f^{-1}(E)$ and $f^{-1}(D)$ have disjoint closures in $T$.

**Proof.** (a)⇒(b). This is clear.

(b)⇒(a). Let $p \in T \setminus X$ and let $\mathcal{F}_p$ be the trace on $X$ of the neighbourhood filter of $p$. In order to prove that there is a continuous extension of $f$, it is sufficient with showing that the filter generated by $f(\mathcal{F}_p) = \{f(F) : F \in \mathcal{F}_p\}$ is a Cauchy filter in $\mu_m Y$. Let $\mathcal{U} \in \mu_m$ arbitrary and consider $\mathcal{V} \in \mu_m$ a star-refinement of $\mathcal{U}$. Since $|\mathcal{V}| < m$ and $X$ is $m$-dense in $T$, there is $V_0 \in \mathcal{V}$ with $f(F) \cap V_0 \neq \emptyset$ for every $F \in \mathcal{F}_p$. If not, let $\mathcal{V} = \{V_i : i \in I\}$ with $|I| < m$; for every $i \in I$ there is $F_i \in \mathcal{F}_p$ such that $f(F_i) \cap V_i = \emptyset$. As $\bigcup \{V_i : i \in I\} = Y$, it follows that $\bigcap \{f(F_i) : i \in I\} = \emptyset$ and, consequently, $\bigcap \{F_i : i \in I\} = \emptyset$, which is a contradiction.

Now let us see that there is $F_0 \in \mathcal{F}_p$ such that $f(F_0) \subseteq \text{st}(V_0, \mathcal{V})$, i.e., $f(F_0)$ is contained in the star of $V_0$ with respect to $\mathcal{V}$. Let us suppose that, for every $F \in \mathcal{F}_p$, $f(F) \cap (Y \setminus \text{st}(V_0, \mathcal{V})) \neq \emptyset$. Then we can define the following two sets: $A = \bigcup \{f(F) \cap V_0 : F \in \mathcal{F}_p\}$ and $B = \bigcup \{f(F) \setminus \text{st}(V_0, \mathcal{V}) : F \in \mathcal{F}_p\}$. We have that $A \subseteq V_0$ and $B \cap \text{st}(V_0, \mathcal{V}) = \emptyset$. This shows that $A$ and $B$ are $\mu$-separated. However, $p \in \text{cl}_T f^{-1}(A) \cap \text{cl}_T f^{-1}(B)$, which is a contradiction.

Thus there exists $F_0 \in \mathcal{F}_p$ with $f(F_0) \subseteq \text{st}(V_0, \mathcal{V})$. Since $\mathcal{V}$ is a star-refinement of $\mathcal{U}$, there is $U_0 \in \mathcal{U}$ such that $f(F_0) \subseteq U_0$. This proves that the filter generated by $f(\mathcal{F}_p)$ is a Cauchy filter in $\mu_m Y$.

**Corollary 2.** Let $\mu X$ be a uniform space. Then $\tau_{\gamma \mu_m} X$ contains the $m$-closure of $\tau\mu X$ in $\tau_{\mu_m} X$.

**Proof.** Let $T$ be the $m$-closure of $\tau\mu X$ in $\tau_{\mu_m} X$, and consider $f : \tau\mu X \rightarrow \tau\mu X$ the identity map. By applying Theorem 1 there is a continuous extension $\bar{f} : T \rightarrow \tau_{\gamma \mu_m} X \subseteq \tau_{\mu_m} X$. Let $g : \tau_{\mu_m} X \rightarrow \tau_{\mu_m} X$ be the identity map and consider $g_{|T} : T \rightarrow \tau_{\mu_m} X$. As $\bar{f}$ and $g_{|T}$ coincide in a dense subspace of $T$, it follows that $\bar{f} = g_{|T}$. Therefore, $\bar{f}$ is the identity map and $T \subseteq \tau_{\gamma \mu_m} X$.

**Corollary 3.** Let $\mu_1$ and $\mu_2$ be two separable (see [8]) uniformities on a space $X$ and suppose that $\tau\mu_i X$ is $Q$-dense ($\mathcal{N}_1$-dense) in $\tau_{\gamma \mu_i} X$, for $i = 1, 2$. Then $\tau_{\gamma \mu_1} X = \tau_{\gamma \mu_2} X$ if and only if for every pair $E$ and $D$ of subsets of $X$ which are $\mu_i$-separated, $\text{cl}_{\tau_{\gamma \mu_1} X} E \cap \text{cl}_{\tau_{\gamma \mu_2} X} D = \emptyset$ for $i \neq j$, $1 \leq i, j \leq 2$.

**Proof.** The proof is clear by taking $f : \tau\mu_i X \rightarrow \tau\mu_j X$ as the identity map in Theorem 1, for $i \neq j$, $1 \leq i, j \leq 2$.

The following result generalizes [5, Theorem 2 and 3, Theorem C].

**Theorem 4.** Let $T$ be a space in which $X$ is dense, let $\mu Y$ be a uniform space, and let $f : X \rightarrow \tau\mu Y$ be a continuous map. Then $f$ has a continuous extension $\bar{f} : T \rightarrow \tau_{\gamma \mu_m} Y$ if and only if

(a) For every pair $E$ and $D$ of subsets of $Y$ which are $\mu$-separated, we have that $\text{cl}_T f^{-1}(E) \cap \text{cl}_T f^{-1}(D) = \emptyset$.

(b) If $\{F_i : i \in I\}$, $|I| < m$, is a family of subsets of $Y$ which has the finite intersection property and $\bigcap \{\text{cl}_{\tau_{\gamma \mu_m} Y} F_i : i \in I\} = \emptyset$, then $\bigcap \{\text{cl}_T f^{-1}(F_i) : i \in I\} = \emptyset$.

**Proof.** The necessity is clear.
Sufficiency. Consider \( f: X \to \tau \mu Y \). By condition (a) we can apply Theorem 1 for \( m = \aleph_0 \). So there is a continuous extension \( \tilde{f} : T - \tau \mu Y \). Now, suppose there is \( p \in T \setminus X \) such that \( \tilde{f}(p) \notin \tau \gamma \mu_m Y \). By Corollary 2, applied to \( \tau \gamma \mu_m Y \), we have that the \( m \)-closure of \( \tau \gamma \mu_m Y \) in \( \tau sU Y \) is \( \tau \gamma \mu_m Y \). Hence there is a family \( \{ V_i : i \in I \} \), \( |I| < m \), of closed neighbourhoods of \( \tilde{f}(p) \) in \( \tau sU Y \) such that \( \bigcap \{ V_i \cap (\tau \gamma \mu_m Y) : i \in I \} = \emptyset \). Consider the family of sets \( \{ V_i \cap (\tau \mu Y) : i \in I \} \). This family verifies the hypotheses of (b). However, \( p \in \bigcap \{ \text{cl} f^{-1}(V_i \cap (\tau \mu Y)) : i \in I \} \) which is a contradiction. This proves that \( f(T) \subseteq \tau \gamma \mu_m Y \).

From Theorem 1 the following result on spaces with bases is derived.

**Theorem 5.** Let \( \mu X \) and \( \nu Y \) be uniform spaces and let \( f: \tau \mu X \to \tau \nu Y \) be a continuous map. Then the following are equivalent:

(a) There is a continuous ext. \( \tilde{f} : \nu(\mu X) \to \tau \nu Y \).

(b) There is a continuous ext. \( \tilde{f} : \nu(\mu X) \to \beta(\nu Y) \).

(c) There is a continuous ext. \( \tilde{f} : \nu(\mu X) \to \nu Y \).

(d) If \( D \) and \( E \) are \( \nu \)-separated in \( Y \), then
\[
\text{cl}_{\nu(\mu X)} f^{-1}(D) \cap \text{cl}_{\nu(\mu X)} f^{-1}(E) = \emptyset.
\]

(e) If \( D \) and \( E \) are \( Z(\nu Y) \)-separated, then
\[
\text{cl}_{\nu(\mu X)} f^{-1}(D) \cap \text{cl}_{\nu(\mu X)} f^{-1}(E) = \emptyset.
\]

(f) If \( D \) and \( E \) are disjoint in \( \nu Y \), then
\[
\text{cl}_{\nu(\mu X)} f^{-1}(D) \cap \text{cl}_{\nu(\mu X)} f^{-1}(E) = \emptyset.
\]

**Proof.** (a)\(\iff\)(d). This is Theorem 1.

(c)\(\iff\)(e). This is a consequence of Theorem 1, by considering in \( Y \) the uniformity associated to the base \( Z(\nu Y) \) (see [7, 6.5(a)]).

(c)\(\iff\)(a), (b)\(\iff\)(f) and (f)\(\iff\)(e). These are clear.

(c)\(\iff\)(a). By [7, Theorem 4.2] \( \nu(\nu Y) \) is the \( Q \)-closure of \( \nu Y \) in \( \tau sU Y \).

(a)\(\iff\)(c). As \( \tau \mu X \) is \( Q \)-dense in \( \nu(\mu X) \), it follows that \( \tilde{f}(\nu(\mu X)) \subseteq \nu Y \).

Let \( X \) be a completely regular Hausdorff space. We denote by \( X' \) the set \( X \) endowed with the \( P \)-topology associated, i.e., the topology for which the collection of \( G_\delta \)-subsets of \( X \) forms an open base. If \( D \) is a base on \( X \), then by \( \sigma(D) \) we mean the \( \sigma \)-algebra of sets generated by \( D \). It is clear that \( \sigma(D) \) is a base on \( X' \).

The next proposition is an application to realcompact spaces of the results above.

**Proposition 6.** Let \( X \) be a realcompact space. If \( \tilde{X} \) is the set \( X \) endowed with a topology such that \( \tilde{X}' = X' \) and, for every pair \( E \) and \( D \) of disjoint Baire sets of \( X \), we have that
\[
\text{cl}_{\nu(\nu(X', \sigma(Z(\tilde{X}))))} E \cap \text{cl}_{\nu(\nu(X', \sigma(Z(\tilde{X}))))} D = \emptyset,
\]
then \( \tilde{X} \) is realcompact.

**Proof.** Let \( f: \tilde{X}' \to X' \) be the identity map. By Theorem 5, there is a continuous extension \( \tilde{f} : \nu(\tilde{X}; \sigma(Z(\tilde{X}))) \to \nu(X'; \sigma(Z(X))) \). Now, by [9, Theorem 16], \( \nu(X', \sigma(Z(X))) = X' \). Then \( \tilde{f}(\nu(X', \sigma(Z(\tilde{X})))) = X' \). Since \( f \) is a homeomorphism, it follows that \( \nu(\tilde{X}', \sigma(Z(\tilde{X}))) = \tilde{X}' \). Therefore \( \tilde{X} \) is realcompact, as a consequence of [9, Theorem 16].

We are informed by the referee that the following result has also been proved by Comfort and Retta in [4].
COROLLARY 7. Let $X$ be a realcompact space, and let $\tilde{X}$ be the set $X$ endowed with a topology finer than the topology of $X$ and such that $\tilde{X}' = X'$. Then $\tilde{X}$ is also realcompact.

PROOF. Since $\sigma(Z(\tilde{X})) \supseteq \sigma(Z(X))$ we can apply Proposition 6.

In [2] Blair proved the following result.

THEOREM 8. Let $X$ be a dense subspace of a topological space $T$, assume each $p \in T \setminus X$ has a countable base of neighbourhoods, let $Y$ be a closed subspace of $R$, and let $f : X \to Y$ be continuous. Then the following are equivalent:

(a) $f$ extends continuously over $T$.

(b) If $F_1$ and $F_2$ are disjoint countable closed subsets of $Y$, then $f^{-1}(F_1)$ and $f^{-1}(F_2)$ have disjoint closures in $T$.

Also Blair proposed in [2] the question of possible generalizations of Theorem 8 for Tychonoff spaces $Y$ that are not necessarily closed subspaces of $R$. What follows is a solution to this question.

THEOREM 9. Let $X$ be a dense subspace of $T$, assume each point $p \in T \setminus X$ has a countable base of neighbourhoods, let $\mu Y$ be a uniform space, and let $f : X \to \tau \mu Y$ be a continuous map. Then the following are equivalent:

(a) $f$ has a continuous extension $\tilde{f} : T \to \tau \gamma \mu Y$.

(b) If $F_1$ and $F_2$ are countable subsets of $Y$ which are $\mu$-separated, then

$$\text{cl}_T f^{-1}(F_1) \cap \text{cl}_T f^{-1}(F_2) = \emptyset.$$
COROLLARY 10. Let $X$ be a dense subspace of $T$, assume each $p \in T \setminus X$ has a countable base of neighbourhoods, let $Y$ be a Tychonoff space, and let $f: X \to Y$ be a continuous map. Then the following are equivalent:

(a) $f$ has a continuous extension $\overline{f}: T \to \nu Y$.

(b) If $F_1$ and $F_2$ are two countable subsets of $Y$ which are completely separated by $C(Y)$, then $\text{cl}_T f^{-1}(F_1) \cap \text{cl}_T f^{-1}(F_2) = \emptyset$.

PROOF. The proof follows from Theorem 9 by considering in $Y$ the weak uniformity generated by the real-valued continuous functions on $Y$.

COROLLARY 11. Let $X$ be a dense subspace of $T$, assume each $p \in T \setminus X$ has a countable base of neighbourhoods, let $Y$ be a first countable Tychonoff space, and let $f: X \to Y$ be a continuous map. Then the following are equivalent:

(a) $f$ has a continuous extension $\overline{f}: T \to Y$.

(b) If $F_1$ and $F_2$ are two disjoint countable closed subsets of $Y$, then $\text{cl}_T f^{-1}(F_1) \cap \text{cl}_T f^{-1}(F_2) = \emptyset$.

PROOF. First we shall prove that if $E$ and $D$ are two subsets of $Y$ which are completely separated, then $\text{cl}_T f^{-1}(E) \cap \text{cl}_T f^{-1}(D) = \emptyset$. Suppose there is $p \in \text{cl}_T f^{-1}(E) \cap \text{cl}_T f^{-1}(D)$ and consider $\{x_n: n \in \mathbb{N}\} \subseteq f^{-1}(E)$, $\{y_n: n \in \mathbb{N}\} \subseteq f^{-1}(D)$ with $\lim x_n = \lim y_n = p$. We are going to define a closed subset $A$ of $Y$ as follows: Take $A = \{f(x_n)\}$ if this set is closed in $Y$. If not, there is $z \in Y$ and a subsequence $\{f(x_{n_k})\}$ of $\{f(x_n)\}$ such that $\lim f(x_{n_k}) = z$. In this case we take $A = \{f(x_{n_k})\} \cup \{z\}$. In the same way we can define a closed subset $B$ of $Y$ from $\{f(y_n)\}$. Thus $A$ and $B$ are two disjoint countable closed subsets of $Y$ with $\text{cl}_T f^{-1}(A) \cap \text{cl}_T f^{-1}(B) = \emptyset$, which is a contradiction. Therefore, $\text{cl}_T f^{-1}(E) \cap \text{cl}_T f^{-1}(D) = \emptyset$.

By Corollary 10 there is a continuous extension $\overline{f}: T \to \nu Y$. Suppose there is $p \in T \setminus X$ with $\overline{f}(p) \in \nu Y \setminus Y$. By hypothesis there is a sequence $\{x_n: n \in \mathbb{N}\} \subseteq X$, with $x_n \neq x_m$ if $n \neq m$, such that $\lim x_n = p$. Let $A = \{f(x_{2n})\}$ and $B = \{f(x_{2n+1})\}$. $A$ and $B$ are disjoint countable closed subsets of $Y$ and $\text{cl}_T f^{-1}(A) \cap \text{cl}_T f^{-1}(B) = \emptyset$. This contradiction proves that $\overline{f}(T) \subseteq Y$.

EXAMPLE 12. In general, the space $\nu Y$ cannot be replaced by $Y$ in Corollary 10.

Let $X$ be a Tychonoff space such that there is a point $p \in \nu X \setminus X$ and a sequence $\{x_n\} \subseteq X$ with $\lim x_n = p$. Let $\tilde{X}$ be the space $X$ but take the point $x_n$ as an open set for all $n \in \mathbb{N}$, and let $T = \tilde{X} \cup \{p\}$, where a base of neighbourhoods for $p$ is given by the sets $V_m = \{x_n: n \geq m\}$, $m \in \mathbb{N}$. If we consider $f: \tilde{X} \to X$ as the identity map, then clearly $f$ verifies condition (b) of Corollary 10. Thus $f$ has a continuous extension $\overline{f}: T \to \nu X$ and $\overline{f}(p) = p \in \nu X \setminus X$.

EXAMPLE 13. The hypothesis of Corollary 11 that $Y$ be a first countable space cannot be relaxed.

Let $X = N$, $T = N^*$ and $Y = \beta N$. If we consider the identity map $f: N \to \beta N$ then, since countable closed sets in $\beta N$ are finite, we have that $f$ verifies condition (b) of Corollary 11. However, $f$ cannot be extended over $T$.

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REFERENCES


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