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CONCERNING THE FUNCTION EQUATION $f(g) = f$, REGULAR MAPPINGS AND PERIODIC MAPPINGS

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ABSTRACT. Under certain conditions on the space X, given $f\colon X \twoheadrightarrow Y$ is light and $g\colon X \twoheadrightarrow X$, the equation $f(g) = f$ yields only periodic solutions for g.

1. Introduction. In [**M**, p. 238], Mioduszewski proved the following theorem: If each of f and g is a mapping of $[0,1]$ onto itself, f is light and $fg = f$, then $g^2 = \mathrm{id}$. Thus g is either the identity or an involution, the only types of periodic maps on $[0,1]$. In this paper we explore more general conditions under which the equation $fg = f$ leads to the conclusion that g is periodic. These results are contained in Theorem 3.1. In §2 we establish some preliminary theorems. §4 contains related examples.

All spaces considered are compact metric and ρ denotes the sup metric on function spaces. A mapping $f\colon X \to Y$ is called *light* iff each point inverse is totally disconnected. A mapping $f\colon X \to X$ is called *positively regular* if the family of iterates $\{f, f^2, f^3, \ldots\}$ is an equicontinuous family. If f is a homeomorphism, then f is *regular* iff the family of all iterates is equicontinuous. Also in case f is a homeomorphism, positively regular implies regular because of compactness. Lemma 2.2 will strengthen this. A mapping $f\colon X \twoheadrightarrow X$ is *periodic* iff there exists a positive integer n such that $f^n = \mathrm{id}$ = the identity mapping on X. The smallest such n is the *period* of f. The double arrow denotes an onto mapping.

2. Preliminaries. In this section we establish the connection between the function equation $fg = f$ and regular homeomorphisms.

LEMMA 2.1. *If g is a positively regular mapping of a compact metric space X onto itself, then either g is a homeomorphism or there exists $\delta > 0$ such that $\rho(g^i, \mathrm{id}) \geq \delta$ for $i = 1, 2, 3, \ldots$.*

PROOF. If g is not a homeomorphism then there exists $c \in X$ such that $g^{-1}(c)$ is nondegenerate. Since the family $\{g, g^2, g^3, \ldots\}$ is equicontinuous, no subsequence of $\mathrm{diam}(g^{-1}(c))$, $\mathrm{diam}(g^{-2}(c))$, $\mathrm{diam}(g^{-3}(c)), \ldots$ converges to 0. It follows that there exists $\delta > 0$ and points $a_i, b_i \in X$ such that $g^i(a_i) = g^i(b_i) = c$ and $d(a_i, b_i) \geq 2\delta$, $i = 1, 2, 3, \ldots$. Then for each i,

$$\begin{aligned}\rho(g^i, \mathrm{id}) &\geq \mathrm{Max}\{d(g^i(a_i), \mathrm{id}(a_i)), d(g^i(b_i), \mathrm{id}(b_i))\}\\ &= \mathrm{Max}\{d(c, a_i), d(c, b_i)\} \geq \delta.\end{aligned}$$

It should be pointed out here that on compact metric spaces, a homeomorphism is regular iff it is almost periodic [**GH**].

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LEMMA 2.2. *If g is a positively regular mapping of a compact metric space X onto itself, then g is a regular homeomorphism.*

PROOF. We can regard the space of all mappings of X onto X as a topological semigroup using the composition operation and the metric ρ. By the Ascoli theorem, $\Gamma(g) = \mathrm{cl}\{g, g^2, g^3, \ldots\}$ is a compact subset and so in fact is a compact subsemigroup. By a theorem of Numakura [**N**, Lemma 3, p. 102], $\Gamma(g) \supset K(g) =$ the set of all cluster points of the sequence $\{g, g^2, g^3, \ldots\}$ is a (an abelian) group. The unit e of $K(g)$ is an idempotent, $e^2 = e$. Since e is a mapping of X onto X, it follows that $e = \mathrm{id}$. And since id is the limit of a subsequence of $\{g, g^2, g^3, \ldots\}$ it follows from Lemma 2.1 that g is a homeomorphism.

LEMMA 2.3. *If each of X and Y is a locally connected compact metric space and $f\colon X \twoheadrightarrow Y$ is a light mapping, then the family of mappings $\{g\colon g\colon X \twoheadrightarrow X$ and $fg = f\}$ is equicontinuous.*

PROOF. Suppose to the contrary. Then there exists $\varepsilon_0 > 0$, $a_i, b_i \in X$ and g_i such that $fg_i = f$, $d(a_i, b_i) < 1/i$ and $d(g_i(a_i), g_i(b_i)) \geq \varepsilon_0$, $i = 1, 2, 3, \ldots$. Wlog assume that there exists $P \in X$ such that $a_i \to P$ and $b_i \to P$. Since X is locally connected at P, it follows that there exists a sequence of continua $A_1 \supset A_2 \supset A_3 \supset \cdots \supset \{P\}$ closing down on P such that, for each i, $\mathrm{diam}(g_i(A_i)) \geq \varepsilon_0$. Again wlog assume that there is a nondegenerate continuum $K \subset X$ such that $g_i(A_i) \to K$.

Now we have $fg_i(A_i) \to f(K)$ but since $fg_i(A_i) = f(A_i) \to f(P)$ it follows that $f(K) = f(P)$. This is not possible since f is light.

THEOREM 2.4. *If each X and Y is a locally connected compact metric space, $f\colon X \twoheadrightarrow Y$ is a light mapping and $g\colon X \twoheadrightarrow X$ is a mapping such that $fg = f$, then g is a regular homeomorphism such that for each $x \in X$ the closure of the g-orbit of x is totally disconnected.*

PROOF. Suppose that X, Y, f and g are given and satisfy the hypothesis. Since for each $i = 1, 2, 3, \ldots, fg^i = f$, it follows from Lemma 2.3 and Lemma 2.2 that g is a regular homeomorphism.

Suppose $x \in X$ and consider $O(x) = \{g^i(x)\colon i = 0, \pm 1, \pm 2, \ldots\}$, the g-orbit of x. We have that, for each integer i, $fg^i(x) = f(x)$ and so $f(\mathrm{cl}(O(x))) = f(x)$. Since f is light, $\mathrm{cl}(O(x)) \subset f^{-1}f(x)$ and must be totally disconnected.

3. The main theorem. The compact metric space G is a *finite graph* iff $V = \{x\colon x \in G$ and order $(x) \neq 2\}$ is finite, $G \sim V$ has finitely many components and if C is a component of $G \sim V$ then $\mathrm{cl}(C)$ is either an arc or a simple closed curve.

Part (1) of Theorem 3.1 establishes a direct generalization of the theorem of Mioduszewski [**M**] referred to in the introduction. Parts (2) and (3) give similar results for some 2-dimensional spaces.

THEOREM 3.1. *Suppose each of X and Y is a metric space, $f\colon X \twoheadrightarrow Y$ is a light mapping and $g\colon X \twoheadrightarrow X$ is a mapping such that $fg = f$. Then g is a periodic homeomorphism if X satisfies one of the following:*

(1) *X is a finite graph.*

(2) *X is a compact subset of the plane whose boundary is a finite graph.*

(3) *X is a compact connected orientable surface.*

PROOF OF PART (1). By Theorem 2.4, g is a regular homeomorphism such that the closure of the orbit of each point is totally disconnected. We will first consider the special case that X is a simple closed curve. There are three possibilities [**H**, p. 129]: (a) If g is order preserving and X has a fixed point under g, then g is the identity on X. (b) If g is order preserving and X has no periodic point under g, then g is topologically an irrational rotation on X. (c) If g is not order preserving, then $g^2 = \mathrm{id}$. Case (b) is eliminated since the orbit of any point in X would be dense in X, and case (c) gives $g^2 = \mathrm{id}$. The remaining case is (a) with some point of period n and here we have $g^n = \mathrm{id}$.

Now for the general case let

$$V = \{x\colon x \in X \text{ and order } (x) \neq 2\}$$

and

$$\mathcal{C} = \{A\colon A \text{ is a component of } X \sim V\}.$$

Both V and $\mathcal{C}$ are finite and the homeomorphism g permutes the elements of V and the elements of $\mathcal{C}$. There exists a positive integer m such that g^m takes each point of V onto itself and each member of $\mathcal{C}$ onto itself. If $A \in \mathcal{C}$, there exists a positive integer $n(A)$ such that $g^{n(A)} = \mathrm{id}$ on $\mathrm{cl}(A)$. This is because the theorem of Mioduszewski [**M**] handles the case that $\mathrm{cl}(A)$ is an arc and the argument given above handles the case that $\mathrm{cl}(A)$ is a simple closed curve. Let n be a common multiple of m and $\{n(A)\colon A \in \mathcal{C}\}$ and then $g^n = \mathrm{id}$ on X.

PROOF OF PART (2). Let us suppose that $\mathrm{int}(X) \neq \varnothing$ since otherwise part (1) would apply. Under the hypothesis, X would be locally connected and so by Theorem 2.4 g is a regular homeomorphism such that the closure of the orbit of each point is totally disconnected. In [**H**, p. 127] it is shown that each component of $\mathrm{int}(X)$ contains a simple closed curve which is invariant under some positive interate of g. Let K be a simple closed curve lying in a component A of $\mathrm{int}(X)$ and m be a positive integer such that $g^m(K) = K$. With the mappings f and g^m restricted to K, $fg^m = f$ and so, by part (1), g^m is periodic on K.

It is also shown in [**H**, p. 129] that if K is a simple closed curve contained in $\mathrm{int}(X)$ and if both K and some point of K are invariant under a positive iterate of g, then g is periodic on the component of X which contains K. Applying this theorem to g^m, we have that g^m is periodic on A and therefore g is periodic on A. For each component A of $\mathrm{int}(X)$ let $n(A)$ be a positive integer such that g restricted to A is of period $n(A)$. Clearly also, g is of period $n(A)$ on $\mathrm{cl}(A)$.

Now $\mathrm{bd}(X)$ is invariant under g and, with the mappings f and g restricted to $\mathrm{bd}(X)$, $fg = f$. By part (1) there is a positive integer l such that g is of period l restricted to $\mathrm{bd}(X)$. Let n be a common multiple of l and $\{n(A)\colon A \text{ is a component of } \mathrm{int}(X)\}$. Then $g^n = \mathrm{id}$ on X.

PROOF OF PART (3). By Theorem 2.4, g is a regular homeomorphism such that the closure of the orbit of each point is totally disconnected. The theorem of von Kerekjarto [**K1**] says that g is periodic if X is anything except a 2-sphere, torus, disc or annulus. If X is a disc or annulus, then part (2) applies (or for a separate argument see [**F** and **R**]).

The regular homeomorphisms on the torus are characterized [**K2**] as being topologically equivalent to a product of standard rotations and such a homeomorphism

would either be periodic, or rotate an invariant simple closed curve K with a dense orbit in K or have a dense orbit. The latter cases cannot occur by Theorem 2.4.

The regular homeomorphisms on the 2-sphere are characterized [**R**] as being topologically equivalent to either a standard rotation or a reflection through the equator or a reflection through the equator followed by a standard rotation. The second type is periodic and each of the other types rotates an invariant simple closed curve [**R**]. In this case, as in the case of the torus, the rotation on the invariant simple closed curve must be a periodic rotation with some period n. The period of g would be n in all cases except for a reflection followed by a rotation on the 2-sphere. In this case the period of g would be n if n is even and $2n$ if n is odd. This completes the proof.

The possibility of extending Theorem 3.1 to include cases of dimension greater than 2 seems difficult since very little is known about regular homeomorphisms in this setting. One approach to the study of regular homeomorphisms in dimension 3 can be found in [**B**].

4. Some examples. If X is a compact metric space and $g\colon X \twoheadrightarrow X$ is periodic, then let Y be the space obtained by decomposing X into g-orbits. If $f\colon X \twoheadrightarrow Y$ is the projection map then of course $fg = f$ and, for any $h\colon Y \twoheadrightarrow Y$, $hfg = hf$. In a similar manner we construct the following

EXAMPLE 4.1. Let X be the locally connected space which is the union of a countably collection of arcs $[a, b_1], [a, b_2], [a, b_3], \ldots$ which have only the point a in common and such that $\operatorname{diam}([a, b_1]) \to 0$ as $i \to \infty$. Let $g\colon X \twoheadrightarrow X$ be the mapping such that g fixes a and permutes the first two arcs $[a, b_1]$ and $[a, b_2]$, permutes the next three arcs with period three, the next four with period four and so on.

Let Y be the space obtained by decomposing X into g-orbits. We see that Y is homeomorphic to X. If $f\colon X \twoheadrightarrow Y$ is the projection map, then $fg = f$. The mapping g is regular and pointwise periodic but not periodic. This example indicates that some sort of finite structure is necessary on X to conclude that g is periodic.

EXAMPLE 4.2. We use X, Y, f and g from Example 4.1. Note that the Hilbert Cube $Q \simeq X \times X \times X \times \cdots$. Let $g^\infty\colon Q \twoheadrightarrow Q$ be the product $g \times g \times g \times \cdots$ and let $f^\infty\colon Q \twoheadrightarrow Q$ be $f \times f \times f \times \cdots$. We have that $fg = f$ and now g is regular but not pointwise periodic.

EXAMPLE 4.3. Let $f\colon [0,1] \twoheadrightarrow [0,1]$ be any mapping which is not light. Let us say that $[a, b]$ is a proper subinterval of $[0,1]$ such that $f([a,b]) = c \in [0,1]$. Then let $g\colon [0,1] \twoheadrightarrow [0,1]$ be any mapping such that $g\colon [a,b] \twoheadrightarrow [a,b]$ and $g(x) = x$ for all $x \notin [a,b]$. Then $fg = f$. This example indicates that the requirement that f be light is necessary to conclude that g is periodic.

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