CONCERNING THE FUNCTION EQUATION $f(g) = f$
REGULAR MAPPINGS AND PERIODIC MAPPINGS

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ABSTRACT. Under certain conditions on the space $X$, given $f: X \to Y$ is light and $g: X \to X$, the equation $f(g) = f$ yields only periodic solutions for $g$.

1. Introduction. In [M, p. 238], Mioduszewski proved the following theorem: If each of $f$ and $g$ is a mapping of $[0,1]$ onto itself, $f$ is light and $fg = f$, then $g^2 = id$. Thus $g$ is either the identity or an involution, the only types of periodic maps on $[0,1]$. In this paper we explore more general conditions under which the equation $fg = f$ leads to the conclusion that $g$ is periodic. These results are contained in Theorem 3.1. In §2 we establish some preliminary theorems. §4 contains related examples.

All spaces considered are compact metric and $\rho$ denotes the sup metric on function spaces. A mapping $f: X \to Y$ is called light iff each point inverse is totally disconnected. A mapping $f: X \to X$ is called positively regular if the family of iterates $\{f, f^2, f^3, \ldots\}$ is an equicontinuous family. If $f$ is a homeomorphism, then $f$ is regular iff the family of all iterates is equicontinuous. Also in case $f$ is a homeomorphism, positively regular implies regular because of compactness. Lemma 2.2 will strengthen this. A mapping $f: X \to X$ is periodic iff there exists a positive integer $n$ such that $f^n = id = \text{the identity mapping on } X$. The smallest such $n$ is the period of $f$. The double arrow denotes an onto mapping.

2. Preliminaries. In this section we establish the connection between the function equation $fg = f$ and regular homeomorphisms.

LEMMA 2.1. If $g$ is a positively regular mapping of a compact metric space $X$ onto itself, then either $g$ is a homeomorphism or there exists $\delta > 0$ such that $\rho(g^i, id) \geq \delta$ for $i = 1, 2, 3, \ldots$.

PROOF. If $g$ is not a homeomorphism then there exists $c \in X$ such that $g^{-1}(c)$ is nondegenerate. Since the family $\{g, g^2, g^3, \ldots\}$ is equicontinuous, no subsequence of $\text{diam}(g^{-1}(c))$, $\text{diam}(g^{-2}(c))$, $\text{diam}(g^{-3}(c))$, \ldots converges to 0. It follows that there exists $\delta > 0$ and points $a_i, b_i \in X$ such that $g^i(a_i) = g^i(b_i) = c$ and $d(a_i, b_i) \geq 2\delta$, $i = 1, 2, 3, \ldots$. Then for each $i$,

$$\rho(g^i, id) \geq \text{Max}\{d(g^i(a_i), id(a_i)), d(g^i(b_i), id(b_i))\} = \text{Max}\{d(c, a_i), d(c, b_i)\} \geq \delta.$$

It should be pointed out here that on compact metric spaces, a homeomorphism is regular iff it is almost periodic [GH].

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LEMMA 2.2. If $g$ is a positively regular mapping of a compact metric space $X$ onto itself, then $g$ is a regular homeomorphism.

PROOF. We can regard the space of all mappings of $X$ onto $X$ as a topological semigroup using the composition operation and the metric $p$. By the Ascoli theorem, $\Gamma(g) = \text{cl}\{g, g^2, g^3, \ldots\}$ is a compact subset and so in fact is a compact subsemigroup. By a theorem of Numakura [N, Lemma 3, p. 102], $\Gamma(g) \supset K(g) =$ the set of all cluster points of the sequence $\{g, g^2, g^3, \ldots\}$ is a (an abelian) group. The unit $e$ of $K(g)$ is an idempotent, $e^2 = e$. Since $e$ is a mapping of $X$ onto $X$, it follows that $e = \text{id}$. And since $\text{id}$ is the limit of a subsequence of $\{g, g^2, g^3, \ldots\}$ it follows from Lemma 2.1 that $g$ is a homeomorphism.

LEMMA 2.3. If each of $X$ and $Y$ is a locally connected compact metric space and $f: X \to Y$ is a light mapping, then the family of mappings $\{g: g: X \to X$ and $fg = f\}$ is equicontinuous.

PROOF. Suppose to the contrary. Then there exists $\varepsilon_0 > 0$, $a_i, b_i \in X$ and $g_i$ such that $fg_i = f$, $d(a_i, b_i) < 1/i$ and $d(g_i(a_i), g_i(b_i)) \geq \varepsilon_0$, $i = 1, 2, 3, \ldots$. Wlog assume that there exists $P \in X$ such that $a_i \to P$ and $b_i \to P$. Since $X$ is locally connected at $P$, it follows that there exists a sequence of continua $A_1 \supset A_2 \supset A_3 \supset \cdots \supset \{P\}$ closing down on $P$ such that, for each $i$, $\text{diam}(g_i(A_i)) \geq \varepsilon_0$. Again wlog assume that there is a nondegenerate continuum $K \subset X$ such that $g_i(A_i) \to K$.

Now we have $fg_i(A_i) \to f(K)$ but since $fg_i(A_i) = f(A_i) \to f(P)$ it follows that $f(K) = f(P)$. This is not possible since $f$ is light.

THEOREM 2.4. If each $X$ and $Y$ is a locally connected compact metric space, $f: X \to Y$ is a light mapping and $g: X \to X$ is a mapping such that $fg = f$, then $g$ is a regular homeomorphism such that for each $x \in X$ the closure of the $g$-orbit of $x$ is totally disconnected.

PROOF. Suppose that $X, Y, f$ and $g$ are given and satisfy the hypothesis. Since for each $i = 1, 2, 3, \ldots, fg^i = f$, it follows from Lemma 2.3 and Lemma 2.2 that $g$ is a regular homeomorphism.

Suppose $x \in X$ and consider $O(x) = \{g^i(x): i = 0, \pm 1, \pm 2, \ldots\}$, the $g$-orbit of $x$. We have that, for each integer $i$, $fg^i(x) = f(x)$ and so $f(\text{cl}(O(x))) = f(x)$. Since $f$ is light, $\text{cl}(O(x)) \subset f^{-1}f(x)$ and must be totally disconnected.

3. The main theorem. The compact metric space $G$ is a finite graph iff $V = \{x: x \in G$ and $\text{order}(x) < 2\}$ is finite, $G \sim V$ has finitely many components and if $C$ is a component of $G \sim V$ then $\text{cl}(C)$ is either an arc or a simple closed curve.

Part (1) of Theorem 3.1 establishes a direct generalization of the theorem of Mioduszewski [M] referred to in the introduction. Parts (2) and (3) give similar results for some 2-dimensional spaces.

THEOREM 3.1. Suppose each of $X$ and $Y$ is a metric space, $f: X \to Y$ is a light mapping and $g: X \to X$ is a mapping such that $fg = f$. Then $g$ is a periodic homeomorphism if $X$ satisfies one of the following:

(1) $X$ is a finite graph.
(2) $X$ is a compact subset of the plane whose boundary is a finite graph.
(3) $X$ is a compact connected orientable surface.
PROOF OF PART (1). By Theorem 2.4, \( g \) is a regular homeomorphism such that the closure of the orbit of each point is totally disconnected. We will first consider the special case that \( X \) is a simple closed curve. There are three possibilities [H, p. 129]: (a) If \( g \) is order preserving and \( X \) has a fixed point under \( g \), then \( g \) is the identity on \( X \). (b) If \( g \) is order preserving and \( X \) has no periodic point under \( g \), then \( g^2 = \text{id} \). Case (b) is eliminated since the orbit of any point in \( X \) would be dense in \( X \), and case (c) gives \( g^2 = \text{id} \). The remaining case is (a) with some point of period \( n \) and here we have \( g^n = \text{id} \).

Now for the general case let

\[
V = \{ x : x \in X \text{ and order } (x) \neq 2 \}
\]

and

\[
\mathcal{C} = \{ A : A \text{ is a component of } X \sim V \}.
\]

Both \( V \) and \( \mathcal{C} \) are finite and the homeomorphism \( g \) permutes the elements of \( V \) and the elements of \( \mathcal{C} \). There exists a positive integer \( m \) such that \( g^m \) takes each point of \( V \) onto itself and each member of \( \mathcal{C} \) onto itself. If \( A \in \mathcal{C} \), there exists a positive integer \( n(A) \) such that \( g^{n(A)} = \text{id} \) on \( \text{cl}(A) \). This is because the theorem of Mioduszewski [M] handles the case that \( \text{cl}(A) \) is an arc and the argument given above handles the case that \( \text{cl}(A) \) is a simple closed curve. Let \( n \) be a common multiple of \( m \) and \( \{ n(A) : A \in \mathcal{C} \} \) and then \( g^n = \text{id} \) on \( X \).

PROOF OF PART (2). Let us suppose that \( \text{int}(X) \neq \emptyset \) since otherwise part (1) would apply. Under the hypothesis, \( X \) would be locally connected and so by Theorem 2.4 \( g \) is a regular homeomorphism such that the closure of the orbit of each point is totally disconnected. In [H, p. 127] it is shown that each component of \( \text{int}(X) \) contains a simple closed curve which is invariant under some positive iterate of \( g \). Let \( K \) be a simple closed curve lying in a component \( A \) of \( \text{int}(X) \) and \( m \) be a positive integer such that \( g^m(K) = K \). With the mappings \( f \) and \( g^m \) restricted to \( K \), \( fg^m = f \) and so, by part (1), \( g^m \) is periodic on \( K \).

It is also shown in [H, p. 129] that if \( K \) is a simple closed curve contained in \( \text{int}(X) \) and if both \( K \) and some point of \( K \) are invariant under a positive iterate of \( g \), then \( g \) is periodic on the component of \( X \) which contains \( K \). Applying this theorem to \( g^m \), we have that \( g^m \) is periodic on \( A \) and therefore \( g \) is periodic on \( A \). For each component \( A \) of \( \text{int}(X) \) let \( n(A) \) be a positive integer such that \( g \) restricted to \( A \) is of period \( n(A) \). Clearly also, \( g \) is of period \( n(A) \) on \( \text{cl}(A) \).

Now \( \text{bd}(X) \) is invariant under \( g \) and, with the mappings \( f \) and \( g \) restricted to \( \text{bd}(X) \), \( fg = f \). By part (1) there is a positive integer \( l \) such that \( g \) is of period \( l \) restricted to \( \text{bd}(X) \). Let \( n \) be a common multiple of \( l \) and \( \{ n(A) : A \text{ is a component of } \text{int}(X) \} \). Then \( g^n = \text{id} \) on \( X \).

PROOF OF PART (3). By Theorem 2.4, \( g \) is a regular homeomorphism such that the closure of the orbit of each point is totally disconnected. The theorem of von Kerekjarto [K1] says that \( g \) is periodic if \( X \) is anything except a 2-sphere, torus, disc or annulus. If \( X \) is a disc or annulus, then part (2) applies (or for a separate argument see [F and R]).

The regular homeomorphisms on the torus are characterized [K2] as being topologically equivalent to a product of standard rotations and such a homeomorphism

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would either be periodic, or rotate an invariant simple closed curve \( K \) with a dense orbit in \( K \) or have a dense orbit. The latter cases cannot occur by Theorem 2.4.

The regular homeomorphisms on the 2-sphere are characterized \([R]\) as being topologically equivalent to either a standard rotation or a reflection through the equator or a reflection through the equator followed by a standard rotation. The second type is periodic and each of the other types rotates an invariant simple closed curve \([R]\). In this case, as in the case of the torus, the rotation on the invariant simple closed curve must be a periodic rotation with some period \( n \). The period of \( g \) would be \( n \) in all cases except for a reflection followed by a rotation on the 2-sphere. In this case the period of \( g \) would be \( n \) if \( n \) is even and \( 2n \) if \( n \) is odd. This completes the proof.

The possibility of extending Theorem 3.1 to include cases of dimension greater than 2 seems difficult since very little is known about regular homeomorphisms in this setting. One approach to the study of regular homeomorphisms in dimension 3 can be found in \([B]\).

4. Some examples. If \( X \) is a compact metric space and \( g: X \to X \) is periodic, then let \( Y \) be the space obtained by decomposing \( X \) into \( g \)-orbits. If \( f: X \to Y \) is the projection map then of course \( fg = f \) and, for any \( h: Y \to Y \), \( hfg = hf \). In a similar manner we construct the following

**Example 4.1.** Let \( X \) be the locally connected space which is the union of a countably collection of arcs \( [a, b_1], [a, b_2], [a, b_3], \ldots \) which have only the point \( a \) in common and such that \( \text{diam}([a, b_i]) \to 0 \) as \( i \to \infty \). Let \( g: X \to X \) be the mapping such that \( g \) fixes \( a \) and permutes the first two arcs \( [a, b_1] \) and \( [a, b_2] \), permutes the next three arcs with period three, the next four with period four and so on.

Let \( Y \) be the space obtained by decomposing \( X \) into \( g \)-orbits. We see that \( Y \) is homeomorphic to \( X \). If \( f: X \to Y \) is the projection map, then \( fg = f \). The mapping \( g \) is regular and pointwise periodic but not periodic. This example indicates that some sort of finite structure is necessary on \( X \) to conclude that \( g \) is periodic.

**Example 4.2.** We use \( X, Y, f \) and \( g \) from Example 4.1. Note that the Hilbert Cube \( Q \cong X \times X \times X \times \cdots \). Let \( g^n: Q \to Q \) be the product \( g \times g \times g \times \cdots \) and let \( f^n: Q \to Q \) be \( f \times f \times f \times \cdots \). We have that \( fg = f \) and now \( g \) is regular but not pointwise periodic.

**Example 4.3.** Let \( f: [0, 1] \to [0, 1] \) be any mapping which is not light. Let us say that \( [a, b] \) is a proper subinterval of \( [0, 1] \) such that \( f([a, b]) = c \in [0, 1] \). Then let \( g: [0, 1] \to [0, 1] \) be any mapping such that \( g: [a, b] \to [a, b] \) and \( g(x) = x \) for all \( x \not\in [a, b] \). Then \( fg = f \). This example indicates that the requirement that \( f \) be light is necessary to conclude that \( g \) is periodic.

**REFERENCES**


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