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THE KREIN-SMULIAN THEOREM

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In this note a brief and accessible proof of the Krein-Smulian theorem will be given.

The lemma below is well known. The map J of the lemma is the canonical embedding of the Banach space X into its second dual X^{**} defined by $Jx(x^*) = x^*(x)$.

LEMMA. *Let X be a separable Banach space and x^{**} a linear functional in X^{**}. Suppose that for each bounded sequence $\{x_n^*\}$ converging in the weak* topology of X^* to x^*, $\{x^{**}(x_n^*)\}$ converges to $x^{**}(x^*)$. Then x^{**} belongs to JX.*

PROOF. Let $\{x_i\}$ be a dense subset of X. Suppose that $d(x^{**}, JX) = d > 0$. By the Hahn-Banach theorem there is a norm one functional x^{***} satisfying $x^{***}(JX) = 0$ and $x^{***}(x^{**}) = d$. Let $W_n = \{z^*\colon |z^*(x_i)| < 1 \text{ for } i = 1, \ldots, n\}$. By Goldstine's theorem there is a functional x^* in the unit ball of X^* for which the values $x^{**}(x^*)$, $x^*(x_1), \ldots, x^*(x_n)$ are as close as desired to the values $x^{***}(x^{**})$, $x^{***}(Jx_1), \ldots, x^{***}(Jx_n)$. Thus there is a functional x_n^* belonging to $\{z^*\colon \|z^*\| \leqslant 1\} \cap \{z^*\colon |x^{**}(z^*)| \geqslant d/2\} \cap W_n$. The sequence $\{x_n^*\}$ converges to zero in the weak* topology, for given x and $\varepsilon > 0$, there is an x_j with $\|(x/\varepsilon) - x_j\| < 1$, and thus $|x_n^*(x)| < 2\varepsilon$ for $n \geqslant j$; yet $|x^{**}(x_n^*)| \geqslant d/2$ for all n. Q.E.D.

THEOREM (KREIN-SMULIAN). *The closed convex hull of a weakly compact subset of a Banach space is weakly compact.*

PROOF. An easy consequence of the Eberlein-Smulian theorem is that it suffices to consider K and therefore X separable. Let K be a weakly compact subset of the separable Banach space X, and let K_w denote K with the weak topology. Consider the map from X^* into the space $C(K_w)$ of continuous functions on K_w defined by $Tx^*(k) = x^*(k)$, and its conjugate $T^*\colon C(K_w)^* \to X^{**}$. Choose any element of $C(K_w)^*$, by the Riesz representation theorem a regular measure μ, and let $\{x_n^*\}$ be a bounded sequence converging in the weak* topology of X^* to x^*. Then $T^*\mu(x_n^*) = \int x_n^*(k)\mu(dk)$ which converges to $\int x^*(k)\mu(dk) = T^*\mu(x^*)$ by the bounded convergence theorem. Thus $T^*\mu$ belongs to JX by the lemma. The unit sphere in $C(K_w)^*$, a weak* compact convex set, is mapped by $J^{-1}T^*$ onto a weakly compact convex set which contains K. Q.E.D.

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Since a weak* convergent sequence is bounded, it is unnecessary but harmless to specify that it be bounded in the lemma. The lemma as given here is a combination of the fact that the weak* topology induced on bounded sets in X^* by a separable X is metrizable [**2**, p. 426] and a more general form of the lemma itself in which X may not be separable, and bounded weak* convergent sequences are replaced by bounded weak* convergent generalized sequences. In this more general version, boundedness is essential, and the lemma is usually proved by means of Dieudonné's powerful BX topology [**2**, p. 428]. The proof given here can easily be modified to give this general result, and so modified it differs from the proof given in [**3**, p. 153] only by the use of Goldstine's theorem instead of Helly's theorem.

Note that the weak and norm closure of a convex set are the same, which takes care of a possible ambiguity in the statement of the theorem.

All the proofs I know for the Krein-Smulian theorem first reduce to the case of separable X, by means of the Eberlein-Smulian theorem [**5**], in order to avoid technical problems. Holmes gives an unusual and interesting proof of the Krein-Smulian theorem using James's theorem that a bounded and weakly closed set is weakly compact if each linear functional attains its supremum there [**3**, p. 162]. The usual approach is to consider the map T^*, or something similar, and then somehow get a map back down to X. In Köthe [**4**, p. 324] this is done using some results concerning the Mackey topology and a lemma due to Grothendieck which is a simple consequence of the bounded convergence theorem. A direct approach is that of Dunford and Schwartz [**2**, p. 434], in which $T^*\mu$ is shown to be in JX by computing its value as J of the Bochner integral $\int k\mu(dk)$. The concise presentation of the vector-valued integration theory necessary for this approach which is given in Diestel [**1**, Chapter IV] can be recommended to any interested reader.

REFERENCES

1. J. Diestel, *Sequences and series in Banach spaces*, Springer-Verlag, New York, 1984.
2. N. Dunford and J. T. Schwartz, *Linear operators*. I, Interscience, New York, 1958.
3. R. Holmes, *Geometric functional analysis and its applications*, Springer-Verlag, New York, 1975.
4. G. Köthe, *Topological vector spaces*. I, Springer-Verlag, New York, 1969.
5. R. Whitley, *An elementary proof of the Eberlein-Smulian theorem*, Math. Ann. **172** (1967), 116–118.

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