A WIENER INVERSION-TYPE THEOREM

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ABSTRACT. Let W(D) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n | ||f||_1 = \sum_{n=0}^{\infty} |a_n| < +\infty\}, f(z) a function in W(D) for which f(0) = 1, and M_f the operator of multiplication by f(z) on W(D). It is shown that if k and m are integers for which 0 \leq m \leq k - 1 and X^m_k is the closed subspace of W(D) spanned by \{z^{nk+i} | n = 0, 1, \ldots; i = 0, 1, \ldots, m\}, then M_f is bounded below on X^m_k if and only if f(z) does not have k - m distinct zeros in any set of the form \{w^i z_0 | 0 \leq i \leq k - 1; |z_0| = 1\}, where w is a primitive kth root of unity.

1. Let W(D) denote the Wiener disc algebra consisting of those analytic functions f(z) = \sum_{n=0}^{\infty} a_n z^n for which ||f||_1 = \sum_{n=0}^{\infty} |a_n| < +\infty. The classical Wiener inversion theorem states that if f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}, where \sum_{n=-\infty}^{\infty} |a_n| < +\infty, then 1/f(\theta) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta} with \sum_{n=-\infty}^{\infty} |b_n| < +\infty \Leftrightarrow f(\theta) \neq 0 for all \theta \[3, p. 196\]. A simple corollary is that if f(z) \in W(D) then multiplication by f(z) on W(D) is an operator M_f which is bounded below \Leftrightarrow f(z) \neq 0 for all |z| = 1. Consequently, a sequence in W(D) of the form \{z^n f(z)\}_{n=0}^{\infty} is a basic sequence which is equivalent to the basis \{z^n\}_{n=0}^{\infty} for W(D) \Leftrightarrow f(z) \neq 0 for all |z| = 1 \[1\].

In \[2\] a result of considerably greater generality was proved. In the context of multiplication operators this result says that if f(z) \in W(D) and k is an integer \geq 1 then the operator M_f of multiplication by f(z) is bounded below on the closed subspace \{z^{nk}\}_{n=0}^{\infty} of W(D) \Leftrightarrow f(z) has no set of k zeros of the form \{w^i z_0\}_{i=0}^{k-1}, where |z_0| = 1 and w is a primitive kth root of unity.

The purpose of this paper is not only to prove a still more general inversion theorem of this type, but to unify the results of \[1 and 2\] mentioned above by fitting them into a context in which they occur as the natural extreme cases of a general theory relating invertibility of M_f to the number of “cyclic zeros” of f(z) on the circle |z| = 1. As a corollary we derive a characterization of certain cyclically shifted sequences in W(D) which are basic and equivalent to the basis \{z^n\}_{n=0}^{\infty}, thereby completing work begun in \[1 and 2\].

2. Throughout the paper f(z) will denote a function in W(D) for which f(0) = 1, k will be a fixed integer \geq 1, and w a primitive kth root of unity.

If m is an integer for which 0 \leq m \leq k - 1 let X^m_k denote the closed subspace of W(D) spanned by the sequence \{z^{nk+i} | n = 0, 1, 2, \ldots; i = 0, 1, \ldots, m\}. Then for any k \geq 1, X^0_k = \{z^{nk}\}_{n=0}^{\infty}, X^{k-1}_k = W(D), and \{X^m_k\}_{m=0}^{k-1} is an increasing tower of subspaces of W(D) between these two “endpoints”. Viewed from this perspective the following theorem simultaneously generalizes and unifies the results of \[1 and 2\].
by describing the invertibility of $M_f$ on the spaces $X_k^m$ in terms of the number of zeros of $f(z)$ lying in sets of the form $\{w^iz_0\}_{i=0}^{k-1}$ on the unit circle. Of particular interest is the way in which this dependence is a monotonic function of $m$.

**Theorem.** The operator $M_f$ is bounded below on $X_k^m$ if and only if $f(z) = 0$ does not have $k - m$ distinct zeros in any set of the form $\{w^iz_0\}_{i=0}^{k-1}$ with $|z_0| = 1$.

**Proof.** Suppose $M_f$ is not bounded below on $X_k^m$. Then there is a sequence $\{g_j(z)\}_{j=1}^{\infty}$ in $X_k^m$ for which $\|g_j\| = 1$ for all $j$, but for which $\|f(z)g_j(z)\| \to 0$. By definition of the space $X_k^m$ each $g_j(z)$ can be written in the form $g_j(z) = \sum_{i=0}^{m} \alpha_i h_i^{(j)}(z)$, where $h_i^{(j)}(z) \in [z^{nk}]_{n=0}^{\infty}$ for all $i$ and $j$, and for which $\sum_{i=0}^{m} \|h_i^{(j)}\|_1 = \|g_j\|_1 = 1$ for all $j$. Since $h_i^{(j)}(z) \in [z^{nk}]_{n=0}^{\infty}$, we have $h_i^{(j)}(w^pz) = h_i^{(j)}(z)$ for all $p$, while from the condition $\sum_{i=0}^{m} \|h_i^{(j)}\|_1 = 1$ we may assume (taking a subsequence of $\{g_j\}$ if necessary) that $\max_{0 \leq i \leq m} \inf_j \|h_i^{(j)}\|_1 > 0$. Therefore, setting $f(z) \cdot g_j(z) = q_j(z)$ for all $j$ and using the observations above together with the assumption that $q_j \to 0$ as $j \to \infty$ we have that

$$
(f(w^l z)) \left( \sum_{i=0}^{m} w^i z^i h_i^{(j)}(z) \right) = q_j(w^l z) \to 0 \quad \text{for all } l = 0, 1, 2, \ldots, m.
$$

Hence

$$
\left( \prod_{i=0}^{m} f(w^i z) \right) \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & w & \cdots & w^m \\
1 & w^2 & \cdots & w^{2m} \\
1 & w^m & \cdots & w^{m^2}
\end{bmatrix} = \begin{bmatrix}
h_0^{(j)}(z) \\
h_1^{(j)}(z) \\
h_2^{(j)}(z) \\
h_m^{(j)}(z)
\end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
$$

Now since $1, w, w^2, \ldots, w^m$ are distinct, the matrix

$$
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & w & \cdots & w^m \\
1 & w^m & \cdots & w^{m^2}
\end{bmatrix}
$$

has a nonzero (Vandermonde) determinant, and hence is invertible. It follows that

$$
\left( \prod_{i=0}^{m} f(w^i z) \right) \begin{bmatrix}
h_0^{(j)}(z) \\
h_1^{(j)}(z) \\
h_2^{(j)}(z) \\
h_m^{(j)}(z)
\end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
$$

and hence that the sequence $\{(\prod_{i=0}^{m} f(w^i z)) \cdot (h_i^{(j)}(z))\}_{j=1}^{\infty}$ converges to zero in $W(D)$ for each $l = 0, 1, \ldots, m$.

In exactly the same way, if $\sigma$ is any subset of $\{0, 1, 2, \ldots, k-1\}$ with cardinality $m+1$ then since $\{w^i\}_{i \in \sigma}$ is again a set of distinct numbers, we have $\prod_{i \in \sigma} f(w^i z) (h_i^{(j)}(z)) \to 0$ in $W(D)$ for each $l = 0, 1, \ldots, m$, where, for some $l,$
inf_j \|h^{(j)}\|_1 > 0$ according to our earlier remarks. Thus $\prod_{i \in \sigma} f(w^i z)|\operatorname{card} \sigma = m + 1$ has a common zero $z_0$ on the unit circle. For, if $A = \{f(\theta) | f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{-i n \theta}, \|f\| = \sum_{n=-\infty}^{\infty} |a_n| < +\infty\}$, then we can identify $W(D)$ with the subspace of $A$ consisting of all such $f(\theta)$ with $a_n = 0$ for $n < 0$. The proper maximal ideals in $A$ are sets of the form $\{f(\theta) | f(\theta_0) = 0\}$ for some $\theta_0 \in [0, 2\pi]$ [3, p. 195], so if $\prod_{i \in \sigma} f(w^i z)|\operatorname{card} \sigma = m + 1$ did not have a common zero on the unit circle it would follow that the corresponding maximal ideal in $A$ generated by this set must be all of $A$. Consequently, there would exist functions $\{r_{\sigma}(\theta)\}$ in $A$ for which $1 = \sum_{\sigma} r_{\sigma}(\theta)(\prod_{k \in \sigma} f(w^k e^{i \theta}))$, and hence for which
\[
h^{(j)}(e^{i \theta}) = \sum_{\sigma} r_{\sigma}(\theta) \prod_{k \in \sigma} f(w^k e^{i \theta})h^{(j)}(e^{i \theta})
\]
for all $l$ and $j$. But by the remarks above this is impossible since the latter sum converges to zero in $A$ as $j \to \infty$, yet $\inf_j \|h^{(j)}\|_1 > 0$. Hence we conclude there must be a number $z_0$ with $|z_0| = 1$ for which $\prod_{i \in \sigma} f(w^i z_0) = 0$ for all subsets $\sigma$ of $\{0, 1, \ldots, k - 1\}$ of cardinality $m + 1$. But then $f(w^i z_0) \neq 0$ for at most $m$ values of $i$, where $0 \leq i \leq k - 1$, so $f(w^i z_0) = 0$ for at least $k - m$ such values of $i$, and we see that $f(z)$ has $k - m$ zeros in the set $\{w^i z_0\}_{i=0}^{k-1}$.

On the other hand, suppose that $f(w^i z_0) = 0$ for some $|z_0| = 1$ and for all $i$ in a subset $\sigma_0$ of $\{0, 1, \ldots, k - 1\}$ of cardinality $k - m$. In particular, suppose $f(z) = \prod_{i \in \sigma_0} (1 - z/w^i z_0)$ (the general case reduces easily to this one, as we show later). Letting $\sigma_0 = \{0, 1, \ldots, k - 1\} \setminus \sigma_0$ we have the cardinality of $\sigma_0 = m$, while
\[
1/f(z) = \prod_{i \in \sigma_0} (1 - z/w^i z_0) = \prod_{i \in \sigma_0} \left(1 - \frac{z}{w^i z_0}\right) = \sum_{n=0}^{\infty} \frac{z^n k}{z_0^n} = \sum_{j=0}^{\infty} b_j z^j,
\]
where
(i) $|b_j| = |b_{nk+j}|$ for all $j$ and all $n$ (since $|z_0| = 1$), and
(ii) $b_j = 0$ for $j \not\in \{nk+i | n = 0, 1, \ldots; i = 0, 1, \ldots, m\}$ (since the degree of $\prod_{i \in \sigma_0} (1 - z/w^i z_0)$ is equal to $m$).

By (ii), for any $r \geq 0$ the polynomial $\sum_{j=0}^{r} b_j z^j \in X_k^m$, while from (i) it follows that for any $\epsilon > 0$ there is a constant $c$ and an integer $r$ for which $\|\sum_{j=0}^{r} c b_j z^j\|_1 = 1$ and $\sup_{0 \leq j \leq r} |c b_j| < \epsilon$. Since $f(z)$ is a polynomial of degree $k - m$ and $1/f(z) = \sum_{j=0}^{\infty} b_j z^j$, for any $r$ the function $f(z) \cdot \sum_{j=0}^{r} b_j z^j$ is equal to $1 + z^{r+1} q_r(z)$, where $q_r(z)$ is some polynomial of degree $k - m - 1$ whose coefficients (for any $r$) are bounded by $L = \sup_j |b_j| \|f(z)\|_1 < +\infty$. Hence, by the above, for any $\epsilon > 0$ we can pick $|c| < \epsilon$ and some $r$ so that $\|\sum_{j=0}^{r} c b_j z^j\|_1 = 1$, but for which $\sup_j |c| |b_j| < \epsilon$, and hence for which
\[
\left\| \left( f(z) \left( \sum_{j=0}^{r} c b_j z^j \right) \right) \right\|_1 \leq |c| \left[ 1 + z^{r+1} q_r(z) \right]_1 \\
\leq |c| \left[ 1 + (k - m) \sup_j |f(z)|_1 \right] \\
\leq |c| + (k - m) \cdot \epsilon \cdot \|f(z)\|_1 < \epsilon [1 + (k - m) \|f(z)\|_1].
\]
That is, $M_f$ is not bounded below on $X_k^m$. 

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For the general case, suppose \( f(z) \) is an arbitrary function in \( W(D) \) for which \( f(w^i z_0) = 0 \) for some \( |z_0| = 1 \) and for all \( i \in \sigma_0 \), a subset of \( \{0, 1, \ldots, k - 1\} \) with cardinality \( k - m \). Since the polynomials are dense in \( W(D) \) and the subspace \( X = \{ g(z) \in W(D) \mid g(w^i z_0) = 0, i \in \sigma_0 \} \) is of finite codimension in \( W(D) \), the polynomials in \( X \) are also dense in \( X \). It follows that \( f(z) \) can be approximated as closely as desired by a polynomial of the form \( q(z) = p_{\sigma_0}(z)r(z) \), where \( p_{\sigma_0}(z) = \prod_{i \in \sigma}(1 - z/w^i z_0) \) (and \( r(z) \) is some other polynomial). Hence if \( M_f \) were bounded below on \( X_k^m \) it would follow that \( M_q \) would also be bounded below on \( X_k^m \) for some such \( q(z) \), an impossibility since we just showed that \( M_{p_{\sigma_0}} \) is not bounded below on \( X_k^m \). Thus if \( f(z) \) has \( k - m \) zeros in the set \( \{ w^i z_0 \mid 0 \leq i \leq k - 1, |z_0| = 1 \} \) then \( M_f \) cannot be bounded below on \( X_k^m \), and the proof is complete.

From this theorem we get the following result which incorporates those of \([1 \text{ and } 2]\) into a unified description of cyclically shifted basic sequences in \( W(D) \).

**Corollary.** If \( f(z) \in W(D) \) with \( f(0) = 1 \), \( k \) is an integer, and \( m \) is an integer for which \( 0 \leq m \leq k - 1 \), then a sequence in \( W(D) \) of the form \( \{ z^{kn+i} f(z) \mid n = 0, 1, \ldots; i = 0, 1, \ldots, m \} \) is a basic sequence equivalent to the basis \( \{ z^n \}_{n=0}^{\infty} \) for \( W(D) \) if \( f(z) \) does not have \( k - m \) distinct zeros in any set of the form \( \{ w^i z_0 \mid 0 \leq i \leq k - 1, |z_0| = 1 \} \).

**References**


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