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ON PERFECT C^*-ALGEBRAS

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ABSTRACT. It is shown that injective C^*-algebras are perfect, as are certain maximal simple C^*-algebras. Some properties of the perfect C^*-algebra A_c arising from a UHF algebra A are obtained by considering a masa with the extension property.

1. Introduction. A notion of perfection for C^*-algebras has been introduced in [**17**] and studied further in [**2**, **1**]. The definition is as follows. Let A be a C^*-algebra, considered as acting in its universal representation so that we may identify the generated von Neumann algebra with A^{**} [**12**, 12.1.3]. Let z be the central projection supporting the atomic part of A^{**}, and let $P(A)$ denote the set of pure states of A. Then A_c is defined to be the set of those elements $b \in zA^{**}$ such that b, b^*b and bb^* are continuous on $P(A) \cup \{0\}$. In fact, A_c is a C^*-subalgebra of zA^{**} [**17**, **2**]. The algebra A is said to be *perfect* if $A_c = zA$.

It has been shown in [**1**] that every separable non-type I C^*-algebra is not perfect. In contrast, we show in Theorem 2.1 that every injective C^*-algebra is perfect. This highlights the importance of the separability assumption in [**1**], since infinite-dimensional injective von Neumann algebras are examples of perfect, nonseparable, non-type I C^*-algebras.

In §3 we show that certain maximal simple C^*-algebras (in the sense of Birrell [**10**]) are perfect. Taken with [**1**, Theorem 3.11], this gives a partial answer to a question raised by Birrell.

In §4 we show that if a C^*-algebra A has a maximal abelian selfadjoint subalgebra (masa) with the extension property then this can be used to give information about A_c. This is illustrated in the case where A is a uniformly hyperfinite (UHF) algebra.

2. Injective C^*-algebras. We recall from [**13**] that a unital C^*-algebra A is said to be *injective* if, given any unital C^*-algebras B and B_1 with $1 \in B \subseteq B_1$ and any unital completely positive linear map $\theta: B \to A$, there exists a completely positive linear map $\theta_1: B_1 \to A$ which extends θ. It follows that injectivity is invariant under $*$-isomorphism. In the special case where $B = A$ and θ is the identity map, θ_1 is a projection of norm one from B_1 onto A.

THEOREM 2.1. *Let A be a unital C^*-algebra which is injective. Then A is perfect.*

PROOF. Since zA is injective there exists a projection of norm one P from A_c onto zA.

Let $\phi \in P(A)$, and let $\overline{\phi}$ denote the normal extension of ϕ to A^{**} and let $\phi^\sim = \overline{\phi}|A_c$. Let $\phi_1(za) = \phi(a)$ $(a \in A)$, so that $\phi_1 \in P(zA)$. Since $\overline{\phi}((1-z)A^{**}) =$

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$\{0\}$, $\phi^{\sim}|zA = \phi_1$. Since the extension of ϕ_1 to a state of A_c is unique [**2**, Proposition 2.9], $\phi^{\sim} = \phi_1 \circ P$.

Let $x \in A_c$. Then $(\phi_1 \circ P)(x) = \phi^{\sim}(x)$ and so $\phi^{\sim}(x - P(x)) = 0$. Since $\{\phi^{\sim}|\phi \in P(A)\}$ is $\sigma(A_c^*, A_c)$ dense in $P(A_c)$ [**2**, Proposition 2.9], $\rho(x - P(x)) = 0$ for all $\rho \in P(A_c)$. Thus $x = P(x) \in zA$.

Since type I von Neumann algebras are injective, Theorem 2.1 generalizes [**2**, Proposition 4.1]. It also follows from Theorem 2.1 that injective factors of type II_1, or type III, on separable Hilbert space, are examples of infinite-dimensional, simple, perfect C^*-algebras. If A is an injective factor of type II_∞ and J is the norm-closed two-sided ideal generated by the finite projections, then by [**2**, Corollary 2.28] J is a (simple) perfect C^*-algebra.

The proof of Theorem 2.1 is adapted below to show that if $P: B \to A$ is a projection of norm one from a C^*-algebra B onto a C^*-subalgebra A then B does not contribute in a nontrivial way to A_c.

PROPOSITION 2.2. *Let B be a unital C^*-algebra and let A be a C^*-subalgebra containing the identity of B. Suppose that there exists a projection of norm one P from B onto A. Then*

$$z_A B z_A \cap A_c = z_A A \quad (\text{in } B^{**})$$

where z_A is the central projection in A^{**} supporting the atomic part of A^{**}.

PROOF. Consider $A \subseteq B \subseteq B^{**}$, acting on the universal Hilbert space for B. Since the identity representation of A is quasi-equivalent to the universal representation of A, we may identify A^{**} with the ultraweak closure of A. Let $\tilde{P} = P^{**}|(z_A B z_A \cap A_c)$. Since P^{**} is a projection of norm one from B^{**} onto A^{**},

$$P^{**}(z_A b z_A) = z_A P(b) z_A \in z_A A \qquad (b \in B).$$

Thus $\tilde{P}$ is a positive linear map of norm one from $z_A B z_A \cap A_c$ onto $z_A A$.

Let $\phi \in P(A)$. We shall use the notation $\phi_1, \phi^{\sim}$ as in the proof of Theorem 2.1. Since $z_A B z_A \cap A_c$ is a selfadjoint subspace of A_c, containing z_A, $\phi^{\sim} \circ \tilde{P}$ extends to a state ψ of A_c [**12**, 2.10.1]. If $a \in z_A A$

$$\psi(a) = (\phi^{\sim} \circ \tilde{P})(a) = \phi^{\sim}(a) = \phi_1(a).$$

Hence $\psi = \phi^{\sim}$ by the uniqueness of extension for ϕ_1.

Let $x \in z_A B z_A \cap A_c$. Then

$$\phi^{\sim}(\tilde{P}(x)) = \psi(x) = \phi^{\sim}(x).$$

As before, $\rho(x - \tilde{P}(x)) = 0$ for all $\rho \in P(A_c)$. Hence $x = \tilde{P}(x) \in z_A A$.

As an example, let B be the Cuntz algebra O_2 and let A be the UHF subalgebra F_2. There is a projection of norm one from O_2 onto F_2 [**11**]. It is known that F_2 is not perfect [**1**], but Proposition 2.2 shows that O_2 does not contribute to this imperfection.

3. Maximal simple C^*-subalgebras. It is shown in [**2**, Corollary 2.22] that if A is a simple C^*-algebra then A_c is simple. This suggests that if A is in some sense maximal simple then perhaps A is perfect. Birrell [**10**] has introduced the notion of a maximal simple C^*-subalgebra of a given C^*-algebra (the existence being guaranteed by an application of Zorn's lemma). In particular, Birrell showed the existence of a maximal simple C^*-subalgebra A of the C^*-algebra $L(H)$ of all bounded linear operators on a certain inseparable Hilbert space H such that A acts irreducibly on H and A is not $*$-isomorphic to any von Neumann algebra.

THEOREM 3.1. *Let A be a C^*-algebra acting irreducibly on a Hilbert space H and suppose that A is a maximal simple C^*-subalgebra of $L(H)$. Then A is perfect.*

PROOF. It is convenient to consider $A \subseteq A^{**}$ and to denote by π the given irreducible representation on H. Let $\bar{\pi}: A^{**} \to L(H)$ be the unique normal extension of π. Then there is a central projection p ($\leq z$) in A^{**} such that $\ker \bar{\pi} = A^{**}(1-p)$, and $xp \to \bar{\pi}(x)$ ($x \in A^{**}$) is a $*$-isomorphism of $A^{**}p$ onto $\overline{\pi(A)}$ ($= L(H)$).

Since $p(zA) = pA \neq \{0\}$ we have $pA_c \neq \{0\}$, and it follows from the simplicity of A_c [**2**, Corollary 2.22] that the map $b \to pb$ ($b \in A_c$) is a $*$-isomorphism of A_c onto pA_c. By the maximality of $\pi(A)$ amongst simple C^*-subalgebras of $L(H)$, we obtain that $\bar{\pi}(pA_c) = \pi(A)$ and hence $pA_c = pA = p(zA)$. Since $zA \subseteq A_c$ and the map $b \to pb$ is one-to-one on A_c, it follows that $A_c = zA$.

The following corollary partly answers a separability question raised by Birrell [**10**, p. 144].

COROLLARY 3.2. *Let A be a C^*-algebra acting irreducibly on a Hilbert space H and suppose that A is a maximal simple C^*-subalgebra of $L(H)$. Then either $A = LC(H)$ (the C^*-algebra of compact linear operators on H) or else A is inseparable.*

PROOF. Suppose $A \neq LC(H)$. Since A is simple, $A \not\supseteq LC(H)$. Hence A is not a type I C^*-algebra. Since A is perfect (by Theorem 3.1), it follows from [**1**, Theorem 3.11] that A must be inseparable.

4. Masas in A and A_c. In this section we show that certain maximal abelian selfadjoint subalgebras (masas) in a C^*-algebra remain maximal when viewed as abelian subalgebras of A_c. In certain cases this can be used to obtain further information about A_c.

We recall that an abelian C^*-subalgebra B of a unital C^*-algebra A is said to have the *extension property* (EP) relative to A [**3**] if each pure state of B has a unique (pure) state extension to A. If B has the (EP) relative to A then, by the Stone-Weierstrass theorem, B is necessarily a masa of A [**15**, **3**]. Furthermore, B has the (EP) relative to A if and only if for each $a \in A$ the set $\overline{\mathrm{co}}^{\mathrm{n}}\{uau^* | u \text{ unitary in } B\}$ has nonempty intersection with B ($\overline{\mathrm{co}}^{\mathrm{n}}$ denotes norm-closed convex hull). When these conditions hold there is a unique projection of norm one P from A onto B such that

$$\{P(a)\} = B \cap \overline{\mathrm{co}}^{\mathrm{n}}\{uau^* | u \text{ unitary in } B\}$$

for each $a \in A$ [**3**, **7**, **9**].

For later use, we also recall now that A is said to have the *Dixmier property* if, for each $a \in A$, $\overline{\mathrm{co}}^{\mathrm{n}}\{uau^* | u \text{ unitary in } A\}$ has nonempty intersection with the centre of A [**6**].

PROPOSITION 4.1. *Let A be a unital C^*-algebra and let B be a masa with the (EP) relative to A. Then zB has the (EP) relative to A_c. In particular, zB is a masa of A_c.*

PROOF. Let $h \in P(zB)$ and let ψ_1, ψ_2 be states of A_c which extend h. Then $\psi_1 | zA$ and $\psi_2 | zA$ are state extensions of h. Since zB has the (EP) relative to

zA, $\psi_1|zA = \psi_2|zA \in P(zA)$. Since any pure state of zA has unique state extension to A_c [**2**, Proposition 2.9], we conclude that $\psi_1 = \psi_2$.

In [**1**, Theorem 3.11] it is shown that any separable non-type I C^*-algebra is not perfect. This is done by first showing that a UHF algebra is not perfect and then using a version of Glimm's reduction [**16**, 6.7.3]. In view of the crucial role played by the UHF algebra in this approach, it seems natural to inquire further as to the nature of A_c for a UHF algebra A (see the example after Proposition 2.2 in this respect). We show below how Proposition 3.1 can be used to gain some information.

Let A be a UHF C^*-algebra. Then A has a masa B with the (EP) relative to A (any 'diagonal' masa has the (EP) relative to A [**4**]). By Proposition 2.1, zB has the (EP) relative to A_c and so there exists a projection of norm one P_c from A_c onto zB such that

$$P_c(x) \in \overline{\mathrm{co}}^{\mathrm{n}}\{uxu^* | u \text{ unitary in } zB\} \tag{1}$$

for each $x \in A_c$.

On the other hand A (and hence zA) has the Dixmier property. This is easy to see directly [**5**, **6**] or may be obtained from the main theorem of [**14**] which states that for a unital simple C^*-algebra the Dixmier property is implied by the existence of at most one tracial state. Coupling this fact with (1) above we see that for $x \in A_c$

$$\overline{\mathrm{co}}^{\mathrm{n}}\{uxu^* | u \text{ unitary in } zA\} \cap \mathbf{C}z \neq \varnothing. \tag{2}$$

Thus A_c enjoys a strong form of the Dixmier property in which it is possible to use unitaries only from zA to effect the required averaging. It follows from (2) by a well-known argument (see, for example, [**6**]) that A_c has at most one tracial state. Indeed, a similar argument shows that A_c has at most one state ψ that is zA-central in the sense that

$$\psi(xa) = \psi(ax) \qquad (a \in zA,\ x \in A_c).$$

Let τ be the unique tracial state of A, and let $\tau_1(za) = \tau(a)$ $(a \in A)$ so that τ_1 is the unique tracial state of zA. Since zA is nuclear it follows that τ_1 is compressible relative to A_c, in the sense of [**8**], and so τ_1 has a zA-central extension ψ in the state space of A_c [**8**, Proposition 3.1]. As shown above, ψ is unique. It follows from (2) that for each $x \in A_c$

$$\{\psi(x)z\} = \overline{\mathrm{co}}^{\mathrm{n}}\{uxu^* | u \text{ unitary in } zA\} \cap \mathbf{C}z.$$

From (1) we obtain that $\psi(P_c(x)) \in \{\psi(x)\}$ for each $x \in A_c$. Thus

$$\psi(x) = \psi(P_c(x)) = \tau_1(P_c(x))$$

and so $\psi = \tau_1 \circ P_c$. It does not seem easy to show directly that $\tau_1 \circ P_c$ is zA-central.

To sum up, $\tau_1 \circ P_c$ extends τ_1 and is the unique zA-central state of A_c. Note that $\tau_1 \circ P_c$ is independent of the choice of B. It would be interesting to know whether or not $\tau_1 \circ P_c$ is a tracial state of A_c. Incidentally, since τ_1 is factorial and ψ is unique, it follows from [**8**, Propositions 3.1 and 2.5] that $\tau_1 \circ P_c$ is factorial.

Finally we remark that if A is a Bunce-Deddens C^*-algebra or an irrational rotation C^*-algebra then the presence of masas with the (EP) enables similar analyses to be undertaken. In particular A_c has a unique zA-central state.

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ADDENDUM. Charles Batty [*Semi-perfect C^*-algebras and the Stone-Weierstrass problem*, preprint] has recently shown that if A is a separable nuclear C^*-algebra then there is a $*$-isomorphism Q_A of A_c into A^{**} such that $Q_A(za) = a$ $(a \in A)$. With A a UHF C^*-algebra as above, let $\bar{\tau}$ be the unique normal extension of τ to A^{**}. Then $\bar{\tau} \circ Q_A$ is a tracial state of A_c, necessarily equal to the state $\tau_1 \circ P_c$ by the uniqueness result above.

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