LINEAR PERTURBATIONS OF
A NONOSCILLATORY SECOND ORDER EQUATION

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ABSTRACT. It is shown that the equation \((r(t)x')' + g(t)x = 0\) has solutions which behave asymptotically like those of a nonoscillatory equation \((r(t)y')' + f(t)y = 0\), provided that a certain integral involving \(f - g\) converges (perhaps conditionally) and satisfies a second condition which has to do with its order of convergence. The result improves upon a theorem of Hartman and Wintner.

We consider the differential equation

\[(1) \quad (r(t)x')' + g(t)x = 0\]

as a perturbation of

\[(2) \quad (r(t)y')' + f(t)y = 0,\]

under the following standing assumption.

ASSUMPTION A. Let \(r\) and \(f\) be real-valued and continuous, with \(r > 0\), on \([a, \infty)\). Suppose that (2) is nonoscillatory at infinity. Let \(g\) be continuous and possibly complex-valued on \([a, \infty)\).

It is known [1, p. 355] that since (2) is nonoscillatory at infinity, it has solutions \(y_1\) and \(y_2\) which are positive on \([b, \infty)\) for some \(b \geq a\) and satisfy the following conditions:

\[(3) \quad r(y_1y'_2 - y'_1y_2) = 1,\]

\[(4) \quad \lim_{t \to \infty} \frac{y_2(t)}{y_1(t)} = \infty.\]

It is convenient to define

\[(5) \quad \rho = \frac{y_2}{y_1}\]
on \([b, \infty)\). From (3) and (4),

\[(6) \quad \rho' = 1/ry_1^2 \quad \text{and} \quad \lim_{t \to \infty} \rho(t) = \infty.\]

Our objective is to extend the following theorem of Hartman and Wintner [1, p. 379].

THEOREM 1 (HARTMAN-WINTNER). Suppose that

\[\int_{y_1y_2|f - g| dt < \infty,}\]

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or, more generally, that
\[ \int_{-\infty}^{\infty} (f-g)y_1^2 \, dt \]
converges (perhaps conditionally), and
\[ \int_{-\infty}^{\infty} \rho' \Gamma \, dt < \infty \]  
(see (6)), where
\[ \Gamma(t) = \sup_{r \geq t} \left| \int_{r}^{\infty} (f-g)y_1^2 \, ds \right| . \]
Then (1) has solutions \( x_1 \) and \( x_2 \) such that \( x_i = (1 + o(1))y_i \) and
\[ \frac{rx_i'}{x_i} = \frac{ry_i'}{y_i} + o\left(\frac{1}{y_1y_2}\right) \]
as \( t \to \infty \), for \( i = 1, 2 \).

The following is our result. After proving it we will show that it is stronger than Theorem 1.

**Theorem 2.** Suppose that
\[ \int_{-\infty}^{\infty} (f-g)y_1y_2 \, dt \]
converges (perhaps conditionally), and
\[ \sup_{r \geq t} \left| \int_{r}^{\infty} (f-g)y_1y_2 \, ds \right| \leq \phi(t), \quad t \geq a, \]
where \( \phi(t) \to 0 \) monotonically as \( t \to \infty \). Define
\[ G(t) = \int_{t}^{\infty} (f-g)y_1^2 \, ds. \]
Now suppose that
\[ \int_{-\infty}^{\infty} \rho' |G| \phi \, dt < \infty \]
and
\[ \lim_{t \to \infty} (\phi(t))^{-1} \int_{t}^{\infty} \rho' |G| \phi \, ds = A < 1/3. \]
Then (1) has a solution \( x_1 \) such that
\[ x_1 = (1 + O(\phi))y_1 \]
and
\[ \frac{(x_1/y_1)'}{x_1/y_1} = O(\phi \rho' / \rho) \]
as \( t \to \infty \), and a solution \( x_2 \) such that
\[ x_2 = (1 + O(\phi_m))y_2 \]
and
\[(x_2/y_2)' = O(\phi_m \rho'/\rho)\]
as \(t \to \infty\), where
\[\phi_m = \max\{\phi, \dot{\phi}\},\]
with
\[\dot{\phi}(t) = \frac{1}{\rho(t)} \int_b^t \rho' \phi \, ds.\]

Notice that integration by parts as in the proof of Abel’s test for convergence of improper integrals shows that if (8) converges, then \(G\) exists and satisfies the inequality
\[|G(t)| \leq 2\phi(t)/\rho(t).\]

We use the contraction mapping theorem [1, p. 404] to establish the existence of \(x_1\). If \(x_1\) satisfies the equation
\[x_1(t) = y_1(t) + \int_t^{\infty} \left[ y_2(s)y_1(t) - y_1(s)y_2(t) \right] (f(s) - g(s))x_1(s) \, ds\]
on \([t_0, \infty)\) for some \(t_0 \geq b\), then \(x_1\) satisfies (1) on \([t_0, \infty)\), and can therefore be continued as a solution of (1) over \([a, \infty)\). Although this suggests an obvious choice of a transformation whose fixed point would be a solution of (1), it is convenient to work instead with a transformation whose fixed point turns out to be the relative error \(z_1 = (x_1 - y_1)/y_1\). Rewriting (20) in terms of \(z_1\) motivates us to consider the transformation \(T\) defined by
\[Tz = Q + Lz,\]
where
\[Q(t) = \int_t^{\infty} \left[ y_2(s) - y_1(s) \rho(t) \right] (f(s) - g(s))y_1(s) \, ds\]
and
\[(Lz)(t) = \int_t^{\infty} \left[ y_2(s) - y_1(s) \rho(t) \right] (f(s) - g(s))y_1(s)z(s) \, ds.\]

We will show that \(T\) is a contraction mapping of a certain Banach space \(B\). It will then be routine to verify that if \(z_1\) is the fixed point of \(T\) in this space and
\[x_1 = y_1(1 + z_1),\]
then \(x_1\) is a solution of (1) which satisfies (13) and (14).

We need the following lemma, which is an elementary extension of Abel’s test. For a proof, see [2, Lemma 1].

**Lemma 1.** Suppose \(u \in C([t_0, \infty))\) and
\[\int_t^{\infty} y_1y_2(f - g)u \, ds\]
converges (perhaps conditionally). Then the function
\[(Lu)(t) = \int_t^{\infty} \left[ y_2(s) - y_1(s) \rho(t) \right] (f(s) - g(s))y_1(s)u(s) \, ds\]
is in $C'(t_0, \infty)$, and it satisfies the inequalities
\[ |(L u)(t)| \leq \sigma(t) \quad \text{and} \quad |(L u)'(t)| \leq 2\sigma(t)\rho'(t)/\rho(t), \quad t \geq t_0, \]
where
\[ \sigma(t) = \sup_{r \geq t} \left| \int_r^\infty y_1y_2(f - g)u \, ds \right|. \]

Since (8) is assumed to converge, Lemma 1 with $u = 1$ implies that
\[ |Q(t)| \leq \phi(t) \]
and
\[ |Q'(t)| \leq 2\phi(t)\rho'(t)/\rho(t). \]
(See (9) and (21).) This motivates us to let $T$ act on the Banach space
\[ B = \{ z \in C'(t_0, \infty) | z = O(\phi), \ z' = O(\phi\rho'/\rho), \ t \to \infty \}, \]
with norm
\[ \|z\| = \sup_{t \geq t_0} \max \left\{ \frac{|z|}{\phi}, \frac{|z'|}{2\phi\rho'} \right\}. \]

We will show that $L$ as defined in (22) is a contraction on $B$ if $t_0$ is sufficiently large. Since (24) and (25) imply that $Q \in B$, it will then follow that $T$ is a contraction on $B$.

Now suppose that $z \in B$, and consider the integral
\[ I(t; z) = \int_t^\infty y_1y_2(f - g)z \, ds. \]
We will show that this integral converges and satisfies the inequality
\[ |I(t; z)| < 3\|z\| m(t)\phi(t), \]
where
\[ m(t) = \phi(t) + (\phi(t))^{-1} \int_t^\infty \rho' |G| \phi \, ds. \]
Then Lemma 1 with $u = z$ will imply that $Lz \in B$ and
\[ \|Lz\| < 3\|z\| \sup_{t \geq t_0} m(t). \]

However, (12) and (29) imply that $\lim_{t \to \infty} m(t) < 1/3$; hence, we see from (30) that $L$ is a contraction if $t_0$ is sufficiently large. Therefore, the proof of existence of $z_1$ is reduced to establishing (28). This is accomplished by rewriting (27) and integrating by parts:
\[ I(t; z) = -\int_t^\infty G'\rho z \, ds \quad \text{(see (5) and (10))} \]
\[ = G(t)\rho(t)z(t) + \int_t^\infty G(\rho z)' \, ds. \]
This integration is valid, since $|G\rho z| \leq 2\|z\|\phi^2$ and $|(\rho z)'| \leq 3\|z\|\phi'$, because of (19) and (26), and the integral on the right converges absolutely, and is dominated by
\[ 3\|z\| \int_t^\infty \rho' |G| \phi \, ds. \]
This implies (28). Hence, $Lz_1 = z_1$ for some $z_1 \in B$. We omit the routine verification that $z_1$ as defined by (23) has the stated properties.

Now we must show that (1) has a solution $x_2$ which satisfies (15) and (16). To this end, choose $b_1 \geq b$ (recall that $y_1, y_2 > 0$ on $[b, \infty)$), so that $x_1$ has no zeros on $[b_1, \infty)$. This is possible, because of (13). We can write
\[ y_2(t) = y_1(t) \left(c + \int_{b_1}^t \frac{ds}{r y_1^2}\right), \quad t \geq b, \]
for a suitable constant $c$. Now define
\[ x_2(t) = x_1(t) \left(c + \int_{b_1}^t \frac{ds}{r x_1^2}\right), \quad t \geq b_1. \]
Then $x_2$ satisfies (1) and, after some manipulations which use (5) and (6), we see that
\[ x_2/y_2 = (x_1/y_1)(1 + \psi), \]
where
\[ \psi(t) = (\rho(t))^{-1} \int_{b_1}^t \rho' \left[ \left(\frac{y_1}{x_1}\right)^2 - 1 \right] \, ds. \]
Because of (13) and definition (18),
\[ \psi = O(\phi). \]
Differentiating (31) and invoking (13), (14), (17), and (32) yields (16). This completes the proof of Theorem 2.

Straightforward manipulations using (5), (6), (13), (14), (15), and (16) show that
\[ rx_1'/x_1 = ry_1'/y_1 + O(\phi/y_1 y_2) \]
and
\[ rx_2'/x_2 = ry_2'/y_2 + O(\phi_m/y_1 y_2) \]
as $t \to \infty$; hence, the conclusions of Theorem 2 imply those of Theorem 1. We will show that the assumptions of Theorem 2 are weaker than those of Theorem 1. Since (7) implies (11) and (12) (with $A = 0$) for any nonincreasing $\phi$, we have only to show that the assumptions of Theorem 1 imply that (8) converges. To this end, we integrate by parts to obtain
\[ \int_{t_1}^{t_2} (f - g)y_1 y_2 \, dt = -G\rho_1^{t_2} + \int_{t_1}^{t_2} \rho' G \, ds. \]
Therefore, because of (7), the convergence of (8) will be established if we show that $\lim_{t \to \infty} G(t)\rho(t) = 0$. If this were not so, there would be a $\gamma > 0$ and an increasing sequence $\{t_j\}$ of points in $[a, \infty)$ such that $\lim_{j \to \infty} t_j = \infty$ and
\[ \Gamma(t_j)\rho(t_j) \geq \gamma, \quad \rho(t_{j-1}) < \rho(t_j)/2, \quad j = 1, 2, \ldots. \]
Then
\[
\int_{t_0}^{\infty} \rho'(s)\Gamma(s) \, ds = \sum_{j=1}^{\infty} \int_{t_{j-1}}^{t_j} \rho'(s)\Gamma(s) \, ds \\
\geq \sum_{j=1}^{\infty} \Gamma(t_j)|\rho(t_j) - \rho(t_{j-1})| > \sum_{j=1}^{\infty} \frac{\gamma}{2} = \infty,
\]
which contradicts (7).

REMARK. If \( \phi(t) \to 0 \) sufficiently slowly by comparison with \( 1/\rho \) as \( t \to \infty \), then \( \phi_m = O(\phi) \). For example, this is true if \( \phi \rho^\mu \) is eventually nondecreasing for some \( \mu < 1 \).

EXAMPLE. Consider the equation
\[
(33) \quad x'' + K[t^{-1}(\log t)^{-\alpha}\sin t]x = 0 \quad (K, \alpha = \text{nonzero constants}),
\]
as a perturbation of \( y'' = 0 \). Then \( y_1 = 1 \), \( y_2 = t \), and
\[
f(t) - g(t) = -Kt^{-1}(\log t)^{-\alpha}\sin t.
\]
Integration by parts shows that if \( \alpha > 0 \) and \( \beta > 0 \), then
\[
(34) \quad \int_{t}^{\infty} s^{-\beta}(\log s)^{-\alpha}\sin s \, ds = t^{-\beta}(\log t)^{-\alpha}(\cos t + O(t^{-1})), \quad t \to \infty.
\]
Now suppose \( \epsilon > 0 \). Then (34) implies that if \( a \) is sufficiently large, then (9) holds with
\[
(35) \quad \phi(t) = (K + \epsilon)(\log t)^{-\alpha}
\]
(take \( \beta = 0 \)), while \( G \) in (10) satisfies the inequality
\[
(36) \quad |G(t)| \leq (K + \epsilon)t^{-1}(\log t)^{-\alpha}, \quad t \geq a
\]
(take \( \beta = 1 \)). Moreover, (34) with \( \beta = 1 \) also implies that
\[
\lim_{t \to \infty} t(\log t)^{a}|G(t)| = K,
\]
which precludes (7) if \( \alpha < 2 \); hence, Theorem 1 does not apply unless \( \alpha \geq 2 \), in which case it implies only that (33) has solutions \( x_1 \) and \( x_2 \) such that \( x_1(t) = 1 + o(1), \ x_1'(t) = o(t^{-1}), \ x_2(t) = t + o(t), \) and \( x_2'(t) = 1 + o(1) \) as \( t \to \infty \). On the other hand, (35) and (36) imply (12) with \( A = K + \epsilon \) if \( \alpha = 1 \), or \( A = 0 \) if \( \alpha > 1 \).
Therefore, Theorem 2 implies that (1) has solutions \( x_1 \) and \( x_2 \) such that
\[
x_1(t) = 1 + O((\log t)^{-\alpha}), \quad x_1'(t) = O(t^{-1}(\log t)^{-\alpha}),
\]
and
\[
x_2(t) = t[1 + O((\log t)^{-\alpha})], \quad x_2'(t) = [1 + O((\log t)^{-\alpha})]
\]
as \( t \to \infty \), provided that either \( \alpha > 1 \) (\( K \) arbitrary) or \( \alpha = 1 \) and \( K < \frac{1}{3} \).

REFERENCES


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